THE TRUNCATED QUEUES WITH BULK ARRIVALS AND HYPEREXPONENTIAL SERVICE TIME DISTRIBUTION WITH RENEGING

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ABSTRACT

This paper deals with the steady-state solution of the queueing system: M^X/H_k /1/N with reneging in which (i) units arrive in batches of random size with the interarrival times of batches following negative exponential distribution, (ii) the batches are served in order of their arrival; and (iii) the service time distribution is hyperexponential with k branches. Recurrence relations connecting the various probabilities introduced are found. Some measures of effectiveness are deduced and some special cases are also obtained.

Keywords: *queueing system: M^X/H_k /1/N, hyperexponential*

DESCRIPTION OF THE SYSTEM

Morse [4] discussed the steady-state queueing system in which the service channel consists of two branches, the units arrive singly and the capacity of the waiting space is infinite. Gupta and Goyal [1] studied a similar system by using the generating functions with k branches in the service channel, the units arrive singly and the capacity of the waiting space is finite. Habib [3] and Gupta and Goyal [2] treated the system $M^X/H_k/1$. White et al. [6] solved the system: M/H_2 /2/2 numerically. All the previous studies are without balking and reneging.

In the present system, it is assumed that the units arrive at the system in batches of random size X, i. e., at each moment of arrival, there is a probability C_j = Pr (X= j) that j units arrive simultaneously, and the interarrival times of batches follow a negative exponential distribution with time independent parameter λ. Let λ C_j Δt, (j = 1, 2,..., N), be the first order probability that a batches of j units comes in time Δt. The service channel is busy if a unit is present in any one of the k branches and in this case the arrival units form a queue and the capacity of the system is N. The unit at the head of the queue requires service in the r<u>th</u> branch with probability σ_r, $\sum_{r=1}^{k} \sigma_r = 1$*. The service time distribution in the r<u>th</u> branch is

* The variation of the subscripts i,j,r,s is from 1 to k, unless otherwise explicitly mentioned.

negative exponential with mean rate μ_r. The overall service-time distribution is of the form

$$S(t) = \sum_r \sigma_r \mu_r e^{-\mu_r t},$$

with mean $\sum_r \frac{\sigma_r}{\mu_r}$.

It is assumed that the units may renege according to an exponential distribution, $f(t) = \alpha e^{-\alpha t}$, $t > 0$, with parameter α. The probability of reneging in a short period of time Δt is given by $r_m = (m-1)\alpha \Delta t$, for $1 < m \le N$ and $r_m = 0$, for $m = 0, 1$.

THE STEADY-STATE EQUATIONS AND THEIR SOLUTION

Define $P_{m,s}$ as the equilibrium probability that there are m units in the system and the unit in the service being in the $s^{\underline{th}}$ branch,

P_0 as the equilibrium probability that there are no units in the system.

Then, the steady–state probability difference equations are

$$\lambda P_0 = \sum_{r=1}^{k} A(1,r) P_{1,r}, \qquad m = 0 \tag{1}$$

$$[\lambda + A(1,s)] P_{1,s} = \sigma_s \lambda C_1 P_0 + \sigma_s \sum_{r=1}^{k} A(2,r) P_{2,r}, \quad m = 1 \tag{2}$$

$$[\lambda + A(m,s)] P_{m,s} = \sigma_s \sum_{r=1}^{k} A(m+1,r) P_{m+1,r} + \lambda \sum_{j=1}^{m-1} C_j P_{m-j,s} + \sigma_s \lambda C_m P_0,$$
$$m = 2(1)N-1 \tag{3}$$

$$[\lambda + A(N,s)] P_{N,s} = \lambda \sum_{j=1}^{N-1} C_j P_{N-j,s} + \lambda \sum_{j=1}^{N} \sum_{i=N-j+1}^{N} C_j P_{i,s} + \sigma_s \lambda C_N P_0,$$
$$m = N \tag{4}$$

where $A(m, s) = \mu_s + (m-1)\alpha$, $m = 1(1)N$, $s = 1(1)k$.

Summing (2) over s and using (1),

$$\sum_r A(2,r) P_{2,r} = \lambda \sum_r P_{1,r} + (1 - C_1) \lambda P_0. \tag{5}$$

Also, summing (3) over s and using (5),

$$\sum_r A(m+1,r) P_{m+1,r} = \lambda \sum_{i=1}^{m} \sum_r P_{i,r} - \lambda \sum_{i=1}^{m-1} \sum_{j=1}^{m-i} \sum_r C_i P_{j,r} - \lambda P_0 \sum_{i=1}^{m} C_i + \lambda P_0,$$
$$2 \le m \le N-1. \tag{6}$$

From (2) and (5),

$$[A(1,r)+\lambda(1-\sigma_r)]P_{1,r}-\lambda\sigma_r\sum_{s\neq r}^{k}P_{1,s}=\sigma_r\lambda P_0, \tag{7}$$

which can be written in the matrix form as:

$$\mathbf{B_1 P_1} = \lambda\, \mathrm{P}_0\, \mathbf{S}, \tag{8}$$

Where

$$\mathbf{B_1} = [\mathrm{b_{ij}}(1)],$$

such that

$\mathrm{b_{ij}}(\mathrm{m}) = -\lambda\sigma_{\mathrm{j}},\ \mathrm{i} \neq \mathrm{j},$

$b_{ii}(m)=A(m,i)+\lambda(1-\sigma_i),$

$\mathbf{P_m^T} = [P_{\mathrm{m},1}, P_{\mathrm{m},2}, \ldots, P_{\mathrm{m,k}}],\ 1 \le m \le N-1$

and

$$\mathrm{S^T} = [\sigma_1, \sigma_2, \ldots, \sigma_k],$$

where T denotes the transpose of a matrix.
Now, the inverse matrix of $\mathbf{B_1}$ is given by

$$\mathbf{B}_1^{-1} = [\mathrm{b}_{ij}^{*}(1)],$$

where

$$b_{ij}^{*}(m)=\frac{\lambda\sigma_i}{[A(m,i)+\lambda][A(m,j)+\lambda]D_m},\ i\neq j$$

$$b_{ii}^{*}(m)=\frac{1}{A(m,i)+\lambda}+\frac{\lambda\sigma_i}{[A(m,i)+\lambda]^2 D_m},$$

such that

$$D_m=1-\sum_{i=1}^{k}\frac{\lambda\sigma_i}{A(m,i)+\lambda},\quad 1\le m\le N.$$

Using this value of $\mathbf{B}_1^{-1}$ in (8), we have

$$P_{1,r}=\frac{\lambda\sigma_r}{(\lambda+\mu_r)D_1}P_0,\quad 1\le r\le k. \tag{9}$$

Similarly, from (3) and (6) at m =2,

$$[A(2,r)+\lambda(1-\sigma_r)]P_{2,r}-\lambda\sigma_r\sum_{s\neq r}^{k}P_{2,s}=\frac{\lambda a_r P_0}{D_1},$$

(10) where

$$a_r=\frac{\sigma_r[(1-C_1)A(1,r)+\lambda]}{A(1,r)+\lambda},$$

which can be written in the matrix form as

$$\mathbf{B_2P_2}=\frac{\lambda\,\mathbf{P_0}}{D_1}\mathbf{A}, (11)$$

where

$$\mathbf{B_2}=\left[\mathrm{b}_{ij}(2)\right],\mathbf{A^T}=[\boldsymbol{a_1},\boldsymbol{a_2},\dots,\boldsymbol{a_k}].$$

Now, the inverse matrix of $\mathbf{B_2}$ is given by

$$\mathbf{B}_2^{-1}=\left[\mathrm{b}_{ij}^{*}(2)\right].$$

Using this value of $\mathbf{B}_2^{-1}$ in (11), we have

$$P_{2,r}=\frac{\lambda P_0}{[A(2,r)+\lambda]D_1}\left\{a_r+\frac{\lambda\sigma_r}{D_2}\sum_{s=1}^{k}\frac{a_s}{A(2,s)+\lambda}\right\}. \tag{12}$$

Similarly, from (3) and (6), we obtain

$$\{A(m,r)+\lambda(1-\sigma_r)\}P_{m,r}-\sigma_r\sum_{s\neq r}^{k}P_{m,s}=\lambda\sum_{j=1}^{m-1}C_j P_{m-j,r}-\sigma_r\eta_m,$$

where

$$\eta_m=\lambda\sum_{i=1}^{m-1}\sum_{j=1}^{m-i}\sum_{s=1}^{k}C_i P_{j,s}-\lambda\sum_{i=1}^{m-1}\sum_{s=1}^{k}P_{i,s}+\lambda P_0\sum_{i=1}^{m-1}C_i-\lambda P_0,\ 3\leq m\leq N-1,$$

which can be written in the matrix form as

$$\mathbf{B_mP_m}=\lambda R_m-\eta_m\,\mathbf{S}, \tag{13}$$

where

$$\mathbf{B_m}=\left[\mathrm{b_{ij}(m)}\right],$$

$$R_m^T=\left[\sum_{j=1}^{m-1}\mathrm{C}_j P_{m-j,1},\sum_{j=1}^{m-1}\mathrm{C}_j P_{m-j,2},\dots,\sum_{j=1}^{m-1}\mathrm{C}_j P_{m-j,k}\right].$$

Now, the inverse matrix of $\mathbf{B_m}$ is given by

$$\mathbf{B_m^{-1}} = \left[\mathrm{b}^*_{\mathrm{ij}}(\mathrm{m})\right].$$

Using this value of $\mathbf{B_m^{-1}}$ in (13), we get

$$P_{m,r} = \frac{\lambda}{[A(m,r)+\lambda]}\left\{\sum_{j=1}^{m-1} C_j P_{m-j,r} + \frac{\lambda\sigma_r}{D_m}\sum_{s=1}^{k}\frac{\sum_{j=1}^{m-1} C_j P_{m-j,s}}{A(m,s)+\lambda}\right\} - \frac{\sigma_r \eta_m}{[A(m,r)+\lambda]D_m},$$

$3 \le m \le N-1.$ (14)

Then, from (4) we have

$$P_{N,r} = \frac{1}{A(N,r)}\left\{\lambda\sum_{j=1}^{N-1} C_j P_{N-j,r} + \lambda\sum_{j=2}^{N}\sum_{i=N-j+1}^{N-1} C_j P_{i,r} + \sigma_r \lambda C_N P_0\right\}, \text{m=N}. \quad (15)$$

Equations (9), (12), (14) and (15) are the required recurrence relations, that give all the probabilities in terms of P_0, which itself may now be determined by using the normalizing condition:

$$P_0 + \sum_{m=1}^{N}\sum_{r=1}^{k} P_{m,r} = 1, \quad (16)$$

Hence all the probabilities are completely known in terms of the queue parameters.

The following example illustrates the method discussed above.

Example:

In the above system: $M^X/H_k/1/N$ with reneging, letting k = 2, N = 4, i.e., the queue: $M^X/H_2/1/4$ with reneging, the results are:

$$P_{1,1} = d_1 P_0, \quad P_{1,2} = d_2 P_0, \quad P_{2,1} = e_1 P_0, \quad P_{2,2} = e_2 P_0,$$
$$P_{3,1} = f_1 P_0, \quad P_{3,2} = f_2 P_0, \quad P_{4,1} = g_1 P_0, \quad P_{4,2} = g_2 P_0,$$

where:

$$d_1 = \frac{\lambda\sigma_1}{(\lambda+\mu_1)D_1}, \ d_2 = \frac{\lambda\sigma_2}{(\lambda+\mu_2)D_1} \qquad \sigma_1+\sigma_2 = 1,$$

$$e_1 = \frac{\lambda}{D_1 D_2(\lambda+\mu_1+\alpha)}\left\{a_1 D_2 + \lambda\sigma_1\left[\frac{a_1}{\lambda+\mu_1+\alpha} + \frac{a_2}{\lambda+\mu_2+\alpha}\right]\right\},$$

$$e_2 = \frac{\lambda}{D_1 D_2(\lambda+\mu_2+\alpha)}\left\{a_2 D_2 + \lambda\sigma_2\left[\frac{a_1}{\lambda+\mu_1+\alpha} + \frac{a_2}{\lambda+\mu_2+\alpha}\right]\right\},$$

$$f_1 = \frac{\lambda}{D_3(\lambda+\mu_1+2\alpha)}\left\{D_3(C_1e_1+C_2d_1)+\lambda\sigma_1\left[\frac{C_1e_1+C_2d_1}{\lambda+\mu_1+2\alpha}+\frac{C_1e_2+C_2d_2}{\lambda+\mu_2+2\alpha}\right]-\frac{Y\sigma_1}{\lambda}\right\},$$

$$f_2 = \frac{\lambda}{D_3(\lambda+\mu_2+2\alpha)}\left\{D_3(C_1e_2+C_2d_2)+\lambda\sigma_2\left[\frac{C_1e_1+C_2d_1}{\lambda+\mu_1+2\alpha}+\frac{C_1e_2+C_2d_2}{\lambda+\mu_2+2\alpha}\right]-\frac{Y\sigma_2}{\lambda}\right\},$$

$$g_1 = \frac{1}{\mu_1+3\alpha}\left\{\lambda\left[d_1(C_3+C_4)+e_1(C_2+C_3+C_4)+f_1\right]+\sigma_1\lambda C_4\right\},$$

$$g_2 = \frac{1}{\mu_2+3\alpha}\left\{\lambda\left[d_2(C_3+C_4)+e_2(C_2+C_3+C_4)+f_2\right]+\sigma_2\lambda C_4\right\},$$

$$a_1 = \sigma_1\left(1-\frac{C_1\mu_1}{\lambda+\mu_1}\right),\ a_2 = \sigma_2\left(1-\frac{C_1\mu_2}{\lambda+\mu_2}\right),$$

$$D_1 = \frac{\lambda\sigma_1\mu_1+\lambda\sigma_2\mu_2+\mu_1\mu_2}{(\lambda+\mu_1)(\lambda+\mu_2)},$$

$$D_2 = \frac{\lambda\sigma_1(\mu_1+\alpha)+\lambda\sigma_2(\mu_2+\alpha)+(\mu_1+\alpha)(\mu_2+\alpha)}{(\lambda+\mu_1+\alpha)(\lambda+\mu_2+\alpha)}$$

$$D_3 = \frac{\lambda\sigma_1(\mu_1+2\alpha)+\lambda\sigma_2(\mu_2+2\alpha)+(\mu_1+2\alpha)(\mu_2+2\alpha)}{(\lambda+\mu_1+2\alpha)(\lambda+\mu_2+2\alpha)},$$

$$Y = \lambda\left\{(C_1-1)(d_1+d_2+e_1+e_2)+C_2(d_1+d_2)\right\}+\lambda(C_1+C_2-1).$$

Now, from (16)

$$P_0+\sum_{m=1}^{4}(P_{m,1}+P_{m,2}) = 1,$$
$$P_0^{-1} = 1+d_1+d_2+e_1+e_2+f_1+f_2+g_1+g_2.$$

Therefore, the expected number of units in the system and in the queue are, respectively,

$$L = \sum_{m=1}^{4}\sum_{r=1}^{2} mP_{m,r} = \sum_{m=1}^{4} m(P_{m,1}+P_{m,2})$$
$$= \left\{d_1+d_2+2(e_1+e_2)+3(f_1+f_2)+4(g_1+g_2)\right\}P_0,$$
$$L_q = \sum_{m=2}^{4}\sum_{r=1}^{2}(m-1)P_{m,r} = \sum_{m=2}^{4}(m-1)(P_{m,1}+P_{m,2})$$
$$= \left\{e_1+e_2+2(f_1+f_2)+3(g_1+g_2)\right\}P_0,$$

and the expected waiting time in both the system and the queue are obtained by

$$W = \frac{L}{\lambda'} \text{ and } W_q = \frac{L_q}{\lambda'},$$

where

$$\lambda^{/} = \frac{\mu}{2}(L - L_q),\ \mu = \mu_1 + \mu_2 .$$

Moreover, if we put $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $\mu_1 = 3$, $\mu_2 = 4$, $\lambda = 5$, $\alpha = 0.3$, $C_1 = 0.4$, $C_2 = 0.3$, $C_3 = 0.1$ and $C_4 = 0.2$, we get:

$P_0 = 0.058901, P_{1,1} = 0.0353436, P_{1,2} = 0.0471248,$
$P_{2,1} = 0.0656447, P_{2,2} = 0.0866124, P_{3,1} = 0.113753,$
$P_{3,2} = 0.149465, P_{4,1} = 0.215968, P_{4,2} = 0.227182.$

Then,
$L = 2.94924$, $L_q = 2.00814$, $W = 0.895383$ and $W_q = 0.609669$.

SPECIAL CASES

Some queueing systems can be obtained as special cases of this system:

(1) Let $\sigma_r = \delta_{rs}$ and $\mu_s = \mu$ where δ_{rs} is the Kronecker delta function, then we get the bulk queue: $M^X/M/1/N$ with reneging, and the results are:

$$P_1 = \rho P_0,\ P_2 = \frac{\rho(\rho + 1 - C_1)}{\delta + 1} P_0,$$

$$P_m = \frac{\rho}{(m-1)\delta + 1}\sum_{j=1}^{m-1} C_j P_{m-j} - \frac{\eta_m^{/}}{(m-1)\delta + 1}, \qquad 3 \le m \le N-1$$

$$P_N = \frac{\rho}{\rho + (N-1)\delta + 1}\left[\sum_{j=1}^{N-1} C_j P_{N-j} + \sum_{j=1}^{N}\sum_{i=N-j+1}^{N} C_j P_i + C_N P_0\right], \text{ m = N}$$

Where

$$\rho = \frac{\lambda}{\mu},\ \delta = \frac{\alpha}{\mu} \text{ and } \eta_m^{/} = \rho\sum_{i=1}^{m-1}\sum_{j=1}^{m-i} C_i P_j - \rho\sum_{i=1}^{m-1} P_i - \rho P_0 \sum_{i=m}^{N} C_i .$$

(2) If we put $C_j = \delta_{j1}$, we get the system: $M/H_k/1/N$ with reneging which studied by Shawky and El-Paoumy [5]. Moreover, if $\alpha = 0$ the system becomes: $M/H_k/1/N$ without reneging which discussed by Gupta and Goyal [1].

(3) Let $N \to \infty$, and $\alpha = 0$, then we have the queue :$M^X/H_k/1$ without reneging which treated by Gupta and Goyal [2] and Habib [3]. Moreover, if $C_j = \delta_{j1}$, $\sigma_r = \delta_{rs}$ and $\mu_s = \mu$, then we get the convenential system: M/M/1.

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