TOTAL CHROMATIC NUMBER OF SPLIT GRAPHS $G = S(I \cup K, E)$ WITH $\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$

Le Xuan Hung

Hanoi University for Natural Resources and Environment
*41 A, Phu Dien Road, Phu Dien Ward, North-Tu Liem District, Hanoi, **VIETNAM***

E-mail: lxhung@hunre.edu.vn

ABSTRACT

A graph $G = (V, E)$ is called a split graph if there exists a partition $V = I \cup K$ such that the subgraphs of G induced by I and K are empty and complete, recpectively. We will denote such a graph by $S(I \cup K, E)$. The notion of split graphs was introduced in 1977 by S. Foldes and P.L. Hammer. These graphs have been paid atthention also because they have connection with packing and knapsack problems, with the matroid theory, with Boolean function. In this paper, some sufficient conditions for split graphs with

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$$

to be Type one are proved.

Keywords: *Split graph; total chromatic number; type one graph; type two graph; chromatic index*

INTRODUCTION

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$ (or V, E in short) will denote its vertex-set, its edge-set, respectively. The set of all neighbours of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denote by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$,the degree of v

(resp., the degree of v with respect to W), denoted by $\deg(v)$, (resp., $\deg_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). The maximum degree of a graph $G=(V,E)$,denoted by $\Delta(G)$ or Δ in short, is the number $\max\{\deg(v)|v\in V\}$. The subgraph of G induced by $W\subseteq V(G)$ is denoted by $G[W]$. The empty and complete graphs of order n are denoted by O_n and K_n. Unless otherwise indicated, our graph-theoretic terminology will follow (M. Behazad and G. Chartrand (1971)).

A graph $G=(V,E)$ is called a *split graph* if there exists a partition $V=I\cup K$ such that $G[I]$ and $G[K]$ are empty and complete graphs, respectively. We will denote such a graph by $S(I\cup K,E)$. The notion of split graphs was introduced in 1977 by Foldes and Hammer (S. Foldes and P.L. Hammer (1977)). A role that split graphs play in graph theory is clarified in (B.-L. Chen et al. (1995); Ngo Dac Tan and Le Xuan Hung (2006)). These graphs have been paid attention also because they have connection with packing and knapsack problems (V. Chvatal and P.L. Hammer (1977)), with the matroid theory (S. Foldes and P.L. Hammer (1978)), with Boolean functions (U.N. Peled (1975)), with the analysis of parallel processes in computer programming (P. B. Henderson and Y. Zalcstein (1977)) and with the task allocation in distributed systems (A. Hesham H. And El-R. Hesham (1993)).

An *edge coloring f* of a graph G is a mapping $f:E(G)\to\{1,2,\ldots\}$ such that two adjacent edges have distinct images. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest integer *k* such that G has an edge coloring having image set $\{1,2,\ldots,k\}$. In 1964, Viring (V.G. Viring (1964)) proved that $\chi'(G)$ is equal to either $\Delta(G)$ or $\Delta(G)+1$. A graph G is said to be *Class one* (resp., *Class two*), if $\chi'(G)=\Delta(G)$ (resp., $\chi'(G)=\Delta(G)+1$).

A *total coloring f* of a graph G is a mapping $f:V(G)\cup E(G)\to\{1,2,\ldots\}$ such that:

(1) No two adjacent vertices or edges have the same image;

(2) The image of each vertex is distinct from the of its incident edges.

The *total chromatic number* of G, denoted by $\chi_T(G)$, is the smallest integer *k* such that G has a total coloring having image set $\{1,2,\ldots,k\}$. Clearly, $\chi_T(G)\geq\Delta(G)+1$. Behzad (M. Behzad (1965)) and Viring (V.G. Viring (1964)) independently made the following conjecture:

Total Coloring Conjecture (TCC). *For any graph* G, $\chi_T(G)\leq\Delta(G)+2$.

If the TCC holds for a certain class of graphs, we shall say that G is *Type one* (resp., *Type two*) if $\chi_T(G)=\Delta(G)+1$ (resp., $\chi_T(G)=\Delta(G)+2$). In 1995, B.-L. Chen, H.-L. Fu and M. T. Ko (B.-L. Chen et al. (1995)) prove that the split graphs satisfy the TCC. In this paper, some sufficient conditions for slit graphs with

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$$

to be Type one are proved.

PRELIMINARIES

We need several lemmas.

Lemma 2.1 (B.-L. Chen et al. (1995)). *Let* $G = S(I \cup K, E)$ *be a split graph with*

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K|.$$

Then G *is Type one.*

Lemma 2.2. *Let* $G = S(I \cup K, E)$ *be a split graph. Then* $\Delta(G) = \deg(v)$ *for some vertex* $v \in K$.

Proof. Since $G[I]$ is a empty graph, we have $\deg(u) \leq |K|$ for any $u \in I$. (1)

Let $\deg(u) = 0$ for any $u \in I$. Then it is not difficult to see that $\Delta(G) = \deg(v)$ for some vertex $v \in K$.

Let there exists $u \in I$ such that $\deg(u) \geq 1$. Then there exists $v_1 \in K$ such that $uv_1 \in E$. It follows that $\deg_I(v_1) \geq 1$. We have

$$\deg(v_1) = \deg_I(v_1) + |K| - 1 \geq |K|. \qquad (2)$$

By (1) and (2), it follows that $\Delta(G) = \deg(v)$ for some vertex $v \in K$. ■

Let $G = S(I \cup K, E)$ be a split graph. Set $G_1 = G - E(G[K])$, it is clear that G_1 is a bipartite graph.

Lemma 2.3. *Let* $G = S(I \cup K, E)$ *be a split graph and* $G_1 = G - E(G[K])$. *Then*

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$$

if and only if $\Delta(G_1) = \deg_I(v)$ *for some vertex* $v \in K$.

Proof. First we prove the necessity. Let $\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$. Suppose that $\deg_I(v) < \Delta(G_1)$ for any $v \in K$. Then $\Delta(G_1) = \max\{\deg(u) \mid u \in I\}$. By Lemma 2.4, there exists $v_1 \in K$ such that $\Delta(G) = \deg(v_1)$. We have

$$\deg(v_1) = \deg_I(v_1) + |K| - 1 < \Delta(G_1) + |K| - 1.$$

It follows that $\Delta(G) < \max\{\deg(u) \mid u \in I\} + |K| - 1$, a contradiction.

Now we the sufficiency. Let $\Delta(G_1) = \deg_I(v)$ for some vertex $v \in K$, say $\Delta(G_1) = \deg_I(v_1)$. It follows that

$$\deg_I(v_1) \geq \max\{\deg(u) \mid u \in I\}.$$

Thus

$$\begin{aligned}\Delta(G) &= \deg_I(v_1) + |K| - 1 \\ &\geq \max\{\deg(u) \| u \in I\} + |K| - 1.\end{aligned}$$

Lemma 2.5 is proved completely. ■

A graph $G = (V, E)$ is called a *complete graph* if all the vertices of G are pairwise adjacent. A complete graph on *n* vertices denote by K_n.

Lemma 2.4 (M. Behazad et al. (1967)). K_n *is Type two if and only if* n *is even.*

A graph $G = (V, E)$ is called a *bipartite graph* if there exists a partition $V = V_1 \cup V_2$ such that the subgraphs of G induced by V_1 and V_2 are empty. We will denote such a graph by $B(V_1 \cup V_2, E)$.

Lemma 2.5 (R.J. Wilson (1975)). *Let* $G = B(V_1 \cup V_2, E)$ *be a bipartite graph. Then*

$$\chi'(G) = \Delta(G).$$

MAIN RESULTS

Let $G = S(I \cup K, E)$ be a split graph. Set $I_K = \{u \in I \mid \deg(u) = |K|\}$.

Theorem 3.1. *Let* $G = S(I \cup K, E)$ *be a split graph with*

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$$

and $|I_K| = 0$. *Then* G *is Type one.*

Proof. If $\Delta(G) > \max\{\deg(u) \mid u \in I\} + |K| - 1$ then by Lemma 2.1, G is Type one. So we may assume that $\Delta(G) = \max\{\deg(u) \mid u \in I\} + |K| - 1$. Set

$$G_1 = G - E(G[K]),\ G_2 = G[K].$$

By Lemma 2.3, $\Delta(G_1)=\deg_I(v)$ for some vertex $v\in K$. Therefore

$$\Delta(G)=\Delta(G_1)+\Delta(G_2).$$

By $|I_K|=0$, $\deg(u)<|K|$ for any $u\in I$. We consider separately two cases.

Case 1: $|K|$ is odd.

By Lemma 2.4, G_2 has a total coloring f_2 using $\Delta(G_2)+1$ colors $1,2,\dots,\Delta(G_2)+1$. The graph G_1 is bipartite. By Lemma 2.5, G_1 has an edge coloring f_1 using $\Delta(G_1)$ colors $\Delta(G_2)+2,\Delta(G_2)+3,\dots,\Delta(G_2)+\Delta(G_1)+1$. Let f be the total coloring of G such that

(i) $f(e)=f_2(e)$ if $e\in E(G_2)$;

(ii) $f(e)=f_1(e)$ if $e\in E(G_1)$;

(iii) $f(v)=f_2(v)$ if $v\in K$;

(iv) $f(u)\in f_2(K)\setminus f_2(N(u))$ if $u\in I$.

Then f is a total coloring of G using $\Delta(G_2)+\Delta(G_1)+1=\Delta(G)+1$ colors. Thus, G is Type one.

Case 2: $|K|$ is even.

By Lemma 2.4, G_2 has a total coloring f_2 using $\Delta(G_2)+2$ colors $1,2,\dots,\Delta(G_2)+2$. The graph G_1 is bipartite. By Lemma 2.5, G_1 has an edge coloring f_1 using $\Delta(G_1)$ colors $\Delta(G_2)+3,\Delta(G_2)+4,\dots,\Delta(G_2)+\Delta(G_1)+2$. Let f be the total coloring of G such that

(i) $f(e)=f_2(e)$ if $e\in E(G_2)$;

(ii) $f(v)=f_2(v)$ if $v\in K$;

(iii) $f(e)=f_1(e)$ if $e\in E(G_1)$ and $f_1(e)\neq\Delta(G_1)+\Delta(G_2)+2$;

(iv) For $e=uv$ with $u\in I, v\in K$ and $f_1(e)=\Delta(G_1)+\Delta(G_2)+2$, $f(e)$ is the color from $\{1,2,\dots,\Delta(G_2)+2\}\setminus\{f_2(v)\}$ not used in the star center v (ie., in the subgraph induced by the edges of G incident with v);

(v) For $u\in I$ and let $v_1,v_2,\dots,v_k$ be the vertices of $N(u)$, $f(u)\in\{1,2,\dots,\Delta(G_2)+2\}\setminus f_2(N(u))$ if

$f(uv_i) \in \{\Delta(G_2)+3, \Delta(G_2)+4, \dots, \Delta(G_2)+\Delta(G_1)+1\}$ for any $i \in \{1,2,\dots,k\}$, $f(u) \in \{1,2,\dots,\Delta(G_2)+2\} \setminus \left(f_2(N(u)) \cup \{f(uv_i)\}\right)$ if there exists $i \in \{1,2,\dots,k\}$ such that $f(uv_i) \in \{1,2,\dots,\Delta(G_2)+2\}$.

Then f is a total coloring of G using $\Delta(G_2)+\Delta(G_1)+1 = \Delta(G)+1$ colors. Thus, G is Type one. ■

Theorem 3.2. *Let $G = S(I \cup K, E)$ be a split graph with*

$$\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$$

and $|I_K| = 1$. Then G is Type one if only if $|V(G)| \geq 3$.

Proof. First we prove the necessity. Suppose that G is Type one. If $|V(G)| < 3$, then it is not difficult to see that $|V(G)| = 2$ and $G \cong K_2$. By Lemma 2.4, K_2 is Type two. So G is Type two, a contradiction.

Now we prove the sufficiency. Suppose that $G = S(I \cup K, E)$ is a split graph with $\Delta(G) \geq \max\{\deg(u) \mid u \in I\} + |K| - 1$, $I_K = \{u_K\}$ and $|V(G)| \geq 3$. It is clear that the graph $G[K \cup \{u_K\}]$ is a complete graph. Let $G' = S(I' \cup K', E')$ be a split graph such that

$$I' = I \setminus \{u_K\}, K' = K \cup \{u_K\}, E' = E.$$

It is not difficult to see that $G' \cong G$, $\max\{\deg(u) | u \in I\} \geq \max\{\deg(u) | u \in I'\} + 1$ and $|K'| = |K| + 1$. It follows that

(i) $\Delta(G') \geq \max\{\deg(u) \mid u \in I'\} + |K'| - 1$;

(ii) $\deg(u) < |K'|$ for any $u \in I'$.

By Theorem 3.1, G' is Type one. So G is Type one. ■

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