THE $M|G|\infty$ QUEUE BUSY PERIOD LENGTH EXPONENTIAL BEHAVIOR FOR PARTICULAR SERVICE TIMES DISTRIBUTIONS COLLECTION

Manuel Alberto M. Ferreira

Instituto Universitário de Lisboa (ISCTE – IUL),
BRU - IUL, ISTAR-IUL Lisboa, ***PORTUGAL***

E-mail: manuel.ferreira@iscte.pt

ABSTRACT

The aim of this work is to look for exponential behavior conditions for the $M|G|\infty$ *queue busy period length distribution, in the case at which a particular service time's collection is considered.*

Keywords: $M|G|\infty$, *busy period, exponential distribution, moments*

Mathematics Subject Classification: *60G99*

INTRODUCTION

In the $M|G|\infty$ queue customers arrive according to a Poisson process at rate λ, upon its arrival receive immediately a service with time length distribution function $G(.)$ and mean α. The traffic intensity is $\rho = \lambda\alpha$. The $M|G|\infty$ queue busy period length distribution is called *B*, the distribution function *B(t)* and the probability density function *b(t)*.

Due to the exponential distribution interesting qualities, in particular the "lack of memory"[1], the pursuit for exponential behavior conditions for the $M|G|\infty$ queue busy period length distribution is done through this work, considering a particular service time's distributions collection.

[1]For instance, if the lifetime of a device is exponentially distributed, the probability that it goes on operational for a period of length T_2, after having been operating for a period of length T_1, is the same that if it had begun its operation:

$$P(T > T_1 + T_2 | T > T_1) = P(T > T_2).$$

That is: the device in T_1 has the same quality as in the beginning of the operation. From here the designation "lack of memory "for this property. This justifies the exponential distribution importance in reliability theory. In fact it is a standard border between the situations $P(T > T_1 + T_2 | T > T_1) \leq P(T > T_2)$ and $P(T > T_1 + T_2 | T > T_1) \geq P(T > T_2)$ where the device incorporates negatively or positively, respectively, the effects of operation time: the memory.

In the next section it is presented a result important for this research. Then in section 3 are presented the aimed results for the M|G|∞ queues, considering a particular service time's distributions collection. This work is finished with the presentation of a conclusions section and a short list of references.

AN IMPORTANT RESULT

The queue busy period length Laplace-Stieltjes transform is, see [5],

$$\bar{B}(s) = 1 + \lambda^{-1}\left(s - \frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t [1-G(v)]dv}dt}\right) \quad (2.1).$$

From (2.1) it is deduced

$$E[B] = \frac{e^{\rho}-1}{\lambda} \quad (2.2),$$

for any service time distribution.

Proposition

With service time distributions satisfying $\lim_{\alpha\to\infty} G(t) = 0$, fixing λ, for α great enough B is approximately exponential.■

Note:

- For this result details see [3].

A SPECIAL COLLECTION OF SERVICE TIMES DISTRIBUTIONS

See now the

Proposition

For an M|G|∞ queue, if the service time distribution function belongs to the collection

$$G(t) = 1 - \frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw - \left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw},$$

$$t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad (3.1)$$

the busy period length distribution function is

$$B(t) = \left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t \beta(u)du} + \lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw\right)\right) *$$

$$* \sum_{n=0}^{\infty} \lambda^n (1-G(0))^n \left(e^{-\lambda t-\int_0^t \beta(u)du}\right)^{*n}, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad (3.2).■$$

Notes:

- The demonstration may be seen in [5],
- For $\frac{\int_0^t \beta(t)dt}{t} = -\lambda$, $G(t) = B(t) = 1, t \geq 0$ in (3.1) and (3.2), respectively,
- For $\frac{\int_0^t \beta(t)dt}{t} = \frac{\lambda}{e^{\rho}-1}$, $B(t) = 1 - e^{-\frac{\lambda}{e^{\rho}-1}t}, t \geq 0$, purely exponential, in (3.2)
- If $\beta(t) = \beta$ (constant)

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}, t\geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1} \tag{3.3}$$

and

$$B^{\beta}(t)=1-\frac{\lambda+\beta}{\lambda}\left(1-e^{-\rho}\right)e^{-e^{-\rho}(\lambda+\beta)t}, t\geq 0, \quad -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1}, \tag{3.4}$$

a mixture of a degenerate distribution at the origin and an exponential distribution,

- With *G(t)* given by (3.1), $\int_0^{\infty}[1 - G(t)]dt =$

$$\int_0^{\infty}\left[\frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^{\infty} e^{-\lambda w-\int_0^w \beta(u)du}dw-\left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw}\right]dt=$$

$-\frac{1}{\lambda}\left[\ln\left|\int_0^{\infty} e^{-\lambda w-\int_0^w \beta(u)du}dw - (1-e^{-\rho})\int_0^t e^{-\lambda w-\int_0^t \beta(u)du}dw\right|\right]_0^{\infty} = \alpha$, such as it should happen, since *B* is positively distributed,

- From (3.1), $\lim_{\alpha\to\infty} G(t) = 0, t \geq 0, -\lambda < \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1}$. Then this service time distributions achieve the Proposition 2.1 conditions. So, the distributions collection (3.2) have approximately exponential behavior for α great enough and $-\lambda < \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1}$,
- For $\beta(t) = \beta$ (constant) it is easy to check directly that the approximately exponential behavior is assumed for α great enough.

To acquire sensitivity to the meaning of α and ρ great enough, it is presented in the sequence the B Coefficient of Variation, $\delta_1[B]$, Coefficient of Symmetry, $\delta_2[B]$ and Kurtosis, $\delta_3[B]$[2] computations for the systems $M|G_1|\infty$[3]. Note that for the $M|G_1|\infty$ queue the $E[B^n], n = 1,2,\ldots,n$ required to compute these parameters are given by

$$E[B^n]=\left(1-e^{-\rho}\right)\frac{n!}{\left(\lambda e^{-\rho}\right)^n}, n=1,2,\ldots \tag{3.5}$$

[2]For the exponential probability distribution $\delta_1 = 1, \delta_2 = 4$ and $\delta_3 = 9$.
[3]Service time distribution given by (3.3) for $\beta = 0$.

Table 1. $\mathbf{M}|\boldsymbol{G_1}|\infty$

Parameters / ρ	$\boldsymbol{\delta_1[B]}$	$\boldsymbol{\delta_2[B]}$	$\boldsymbol{\delta_3[B]}$
.5	2.0206405	9.5577742	15.983720
1	1.4710382	5.5867425	10.878212
10	1.0000454	4.0000000	9.0000000
20	1.0000000	4.0000000	9.0000000
50	1.0000000	4.0000000	9.0000000
100	1.0000000	4.0000000	9.0000000

The results in Table 3.1 clearly show that after $\rho = 10$ the $\mathrm{M}|G_1|\infty$ queue busy period length exponential behavior is evidenced.

CONCLUSIONS

The exponential distribution is very simple and quite useful from a practical point of view. It has been frequently considered in queuing systems study. Conditions under which B is exponentially distributed or approximately exponentially distributed for the $\mathrm{M}|\mathrm{G}|\infty$ queue may be seen in [3].

Many quantities of interest in queues are insensible. This means that they depend on the service time distribution only by its mean. Thus it is indifferent which service distribution is being considered. But using those given by (3.3), result quasi-exponential or exponential busy periods. And, for these service distributions, all distributions related to the busy period have simple forms and are related to the exponential distribution.

For the distributions collection given by (3.1) it was seen that under conditions of heavy- traffic -α great- B is approximately exponential.

REFERENCES

1) L. Tackács. An Introduction to Queueing Theory. Oxford University Press. New York, 1962.
2) J. A. Filipe, M. A. M. Ferreira. Infinite Servers Queue Systems Busy Period-A Practical Case in Logistics Problems Solving. Applied Mathematical Sciences, 9(2015), 25-28, 1221-1228.
3) M. A. M. Ferreira. The Exponentiality of the queue busy period. XII Jornadas Luso-Espanholas de Gestão Científica-ACTAS (Volume VIII-Economia da Empresa e Matemática Aplicada), 267-272. Universidade da Beira Interior, Departamento de Gestão e Economia, Covilhã, Abril 2002.
4) M. A. M. Ferreira, J. A. Filipe. Occupation Study of a Surgery Service through a Queues Model. Applied Mathematical Sciences, 9(2015), 93-96, 4791-4798.

5) M. A. M. Ferreira, M. Andrade. The Ties between the $M|G|\infty$ Queue System Transient Behavior and the Busy Period. International Journal of Academic Research, 1 (2009), 1, 84-92.
6) M. A. M. Ferreira, M. Andrade. Looking to a $M|G|\infty$ system occupation through a Riccati equation. Journal of Mathematics and Technology, 1(2010), 2, 58-62.
7) M. A. M. Ferreira, M. Andrade. $M|G|\infty$, System Transient Behavior with Time Origin at the Beginning of a Busy Period Mean and Variance. Aplimat- Journal of Applied Mathematics, 3(2010), 3, 213-221.
8) M. A. M. Ferreira, M. Andrade. $M|G|\infty$ Queue Busy Period Tail. Journal of Mathematics and Technology, 1(2010), 3, 11-16.
9) M. A. M. Ferreira, M. Andrade. Infinite Servers Queue Systems Busy Period Length Momemts. Journal of Mathematics and Technology, 3(2012), 2, 4-7.
10) M. A. M. Ferreira, M. Andrade, J. A Filipe. The Riccati Equation in the $M|G|\infty$ System Busy Cycle Study. Journal of Mathematics, Statistics and Allied Fields, 2 (2008), 1.
11) M. J. Carrillo. Extensions of Palm's Theorem: A Review. Management Science, 37(1991), 6, 739-744.
12) W. Stadje. The Busy Period of the Queueing System $M|G|\infty$. Journal of Applied Probability, 22(1985), 697-704.