SOME CHARACTERIZATION FOR RECORD VALUES

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ABSTRACT

In this paper we discussed some explicit forms for conditional expectations of record values. We study the necessary and sufficient conditions such that the conditional expectations of record values hold for some distribution functions.

Keywords: *Conditional expectation, characterization of distribution, record values*

INTRODUCTION

Record values play an important role in many aspects of daily life as well as in many statistical applications. Often the interest is focused in observing new records and in recording them: for example, world or Olympic records in sport.

Record values are used in reliability theory. Furthermore, these statistics are closely connected with the occurrences times of some corresponding non homogeneous Poisson process used in shock models. The statistical study of record values started with Chandler [5], the theory of record values was formulated as a model for successive extremes in a sequence of independently and identically random variables. Some examples of record values were given in Feller [7] with respect to gambling problems. The asymptotic theory of records was discussed in Resnick [21]. Theory of record values and its distributional properties have been broadly studied in the literature, for example, see, Ahsanullah [2], Arnold *et al*. [3], Nevzorov [18] and Kamps [12] for reviews on various developments in the area of records and references therein.

Suppose that $X_1, X_2, \dots$ is a sequence of independent and identically distributed (i.i.d.) random variables with distribution function (df) F(x) and probability density function (pdf) f(x). Let Y_n = max(min) $\{X_1, X_2, \dots, X_n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}, j > 1$. By definition, X_1 is a lower as well as an upper record value. The upper records can be transformed to lower records by replacing the original sequence of $\{X_j\}$ by $\{-X_j, j \geq 1\}$ or (if $P(X_j > 0) = 1$ for all j) by $\{1/X_j, j \geq 1\}$; the lower record values of this sequence will correspond to the upper record values of the original sequence.

Let $Y_{L(1)}, Y_{L(2)}, \dots, Y_{L(n)}$ be the first n lower record values from a population whose pdf f(x) and cdf F(x). Then, the pdf of the $Y_{L(m)}, m = 1, 2, \dots$ is given by (see Ahsanullah [2] and Arnold et al. [3])

$$f_{X_{L_{(m)}}}(x) = \frac{1}{\Gamma(m)}\{-logF(x)\}^{m-1} f(x), m > 1, -\infty < x < \infty, \tag{1}$$

where $\Gamma(.)$ is a gamma function, and the joint pdf of $X_{L_{(m)}}$ and $X_{L_{(n)}}$, $(1 \le m < n)$, is given by

$$f_{Y_{L_{(m)}}, Y_{L_{(n)}}}(x, y) = \frac{1}{\Gamma(m)\,\Gamma(n-m)}\{-logF(x)\}^{m-1}\{logF(x) - logF(y)\}^{n-m-1} \times \frac{f(x)}{F(x)} f(y), -\infty < y < x < \infty. \tag{2}$$

Now, let $Z_{U(1)}, Z_{U(2)}, \dots, Z_{U(n)}$ be the first n upper record values, then, the pdf and the joint pdf of $Z_{U(m)}$, and $(Z_{U(m)}, Z_{U(n)})$, $m < n$ respectively, are given by (see Ahsanullah [2] and Arnold et al. [3])

$$f_{Z_{U(m)}}(x) = \frac{1}{\Gamma(m)}\left\{-log\overline{F}(x)\right\}^{m-1} f(x), m > 1, -\infty < x < \infty, \tag{3}$$

$$f_{Z_{U(m)}, Z_{U(n)}}(x, y) = \frac{1}{\Gamma(m)\,\Gamma(n-m)}\left\{-log\overline{F}(x)\right\}^{m-1}\left\{log\overline{F}(x) - log\overline{F}(y)\right\}^{n-m-1} \times \frac{f(x)}{\overline{F}(x)} f(y), -\infty < x < y < \infty. \tag{4}$$

Characterization of distributions through conditional expectation of record values have been considered among others by Nagaraja [17], Franco and Ruiz [8, 9], Lopez-Blazquez and Moreno-Rebollo [15], Abu-Youssef [1], Dembinska and Wesolowski [6], Raqab [20], Athar et al. [4], Khan and Alzaid [13] and Wu [22]. Khan et al. [14] characterized a family of continuous distributions through difference of two conditional expectations of record values. Malinowska and Szynal [16] assume that the common random variable X is an absolutely continuous random variable concentrated on the interval (α, β) with $F(x) < 1$ for $x \in (\alpha, \beta)$, $F(\alpha) = 0$ and $F(\beta) = 1$.

For a given monotonic and differentiable function g on (α, β), we write

$$E[g(X)|X \le x] = \frac{1}{F(x)}\int_{\alpha}^{x} g(u)\, f(u)\, du, \tag{5}$$

$$E[g(X)|X \ge x] = \frac{1}{\overline{F}(x)}\int_{x}^{\beta} g(u)\, f(u)\, du, \tag{6}$$

$$F(x) = \left[a + b\, e^{-c\, g(x)}\right]^{d}, \quad b, c, d \ne 0, \tag{7}$$

$$\overline{F}(x) = \left[a + b\, e^{-c\, g(x)}\right]^{d}, \; b, c, d \ne 0. \tag{8}$$

In this paper, we have characterized two general forms of distributions through conditional expectation of p-th power of difference of functions of two record values (Theorem 1 on lower record values and Theorem 2 on upper record values).

Theorem 1:

Consider X to be an absolutely continuous random variable with cdf F(x) and probability density function (pdf) f(x) on the support (α, β). Then, for two lower record values of m and n,

$$E\left(\{g(Y_{L(n)}) - g(Y_{L(m)})\}^p \middle| Y_{L(m)} = x\right) = \xi_{m,n,p} = \frac{\Gamma(p+n-m)}{(c)^p\,\Gamma(n-m)} \quad for\ \alpha < y < x < \beta, \tag{9}$$

If and only if

$$F(x) = b\, e^{-c\,\varphi(x)},\ \ b, c \neq 0. \tag{10}$$

Proof

To prove the necessary part, from (1) and (2), we have for $n \geq m + 1$,

$$\xi_{m,n,P} = \frac{1}{\Gamma(n-m)} \int_\alpha^x \{g(y) - g(x)\}^p \left\{log\left(\frac{F(x)}{F(y)}\right)\right\}^{n-m-1} \frac{f(y)}{F(x)}\, dy\,. \tag{11}$$

Let, using (10)

$$w = g(y) - g(x),\ \ \{g(y) - g(x)\}^p = w^p$$

$$\xi_{m,n,P} = \frac{c^{n-m}}{\Gamma(n-m)} \int_0^\infty W^{p+n-m-1}\ e^{-cw}\, dw\ = \frac{\Gamma(p+n-m)}{c^P\,\Gamma(n-m)}.$$

Then the necessary condition is proved, to prove the sufficiency condition, it is clear from (11) that

$$F(x)\,\xi_{n,m,P} = \frac{1}{\Gamma(n-m)} \int_\alpha^x \{g(y) - g(x)\}^p \left\{log\frac{F(x)}{F(y)}\right\}^{n-m-1} f(y)dy. \tag{12}$$

Differentiating both sides in (12) with respect to x, we get

$$f(x)\,\xi_{m,n,p} = \frac{(n-m-1)\,f(x)}{\Gamma(n-m)} \int_\alpha^x \{g(y) - g(x)\}^p \left\{log\left(\frac{F(x)}{F(y)}\right)\right\}^{n-m-2} \frac{f(y)}{\overline{\overline{F(x)}}}\, dy$$
$$- \frac{p\,\acute{g}(x)\,F(x)}{\Gamma(n-m)} \int_\alpha^x \{g(y) - g(x)\}^{p-1} \left\{log\left(\frac{F(x)}{F(y)}\right)\right\}^{n-m-1} \frac{f(y)}{\overline{\overline{F(x)}}}\, dy.$$

We can obtain (using 11) that

$$f(x)\,\xi_{m,n,p} = f(x)\,\xi_{m,n-1,P} - p\,\acute{g}(x)\,F(x)\,\xi_{m,n,p-1}$$

and from (9)

$$p\,\acute{g}(x)\,\xi_{m,n,p-1} = \frac{f(x)}{F(x)}\left(\xi_{m,n-1,p} - \xi_{m,n,p}\right)$$
$$= -\frac{p\,f(x)}{F(x)}\,\frac{\Gamma(p+n-m-1)}{c^p\,\Gamma(n-m)} = -\frac{p\,f(x)}{c\,F(x)}\ \ \xi_{m,n,p-1},$$

which implies

$$-c\,\acute{g}(x) = \frac{f(x)}{F(x)}, \tag{13}$$

hence, the theorem is proved.

The case when b = 1, which is the special case of theorem 1, have been treated in Noor and Athar [19]

Theorem 2:

Consider X to be an absolutely continuous random variable with distribution function F(x) and probability density function f(x) on the support (α,β). Then, for two values of m and n,

$$E\left(\left\{g\left(X_{U_{(n)}}\right)-g\left(X_{U_{(m)}}\right)\right\}^{p}\middle|X_{U_{(m)}}=x\right)=\xi_{m,n,p}=\frac{\Gamma(p+m-n)}{c^{p}\Gamma(n-m)}\quad for\ \alpha<x<y<\beta, \tag{14}$$

if and only if

$$\overline{F(x)}=b\,e^{-c\,g(x)},\ \ b,c\neq 0. \tag{15}$$

Proof

To prove the necessary part, from (3) and (4) we have for n≥ m+1

$$\xi_{m,n,p}=\frac{1}{\Gamma(n-m)}\int_{x}^{\beta}\{g(y)-g(x)\}^{p}\left\{ln\left(\frac{\overline{F(x)}}{\overline{F(y)}}\right)\right\}^{s-r-1}\frac{f(y)}{\overline{F(x)}}\,dy. \tag{16}$$

Let (using (15))

$$w=ln\left(\frac{\overline{F(x)}}{\overline{F(y)}}\right)\ ,\ \ \{g(y)-g(x)\}^{p}=\left(\frac{w}{c}\right)^{p}$$

$$\xi_{r,s,p}=\frac{1}{\Gamma(n-m)}\int_{0}^{\infty}\left(\frac{w}{c}\right)^{p}w^{n-m-1}e^{-w}\,dw=\frac{\Gamma(p+n-m)}{b^{p}\Gamma(n-m)},$$

this proves the necessary condition. To prove the sufficiency condition, it is clear from (16) that

$$\overline{F(x)}\xi_{m,n,p}=\frac{1}{\Gamma(n-m)}\int_{x}^{\beta}\{g(y)-g(x)\}^{p}\left\{ln\left(\frac{\overline{F(x)}}{\overline{F(y)}}\right)\right\}^{n-m-1}f(y)\,dy. \tag{17}$$

After differentiating both sides in (17) with respect to x, we get

$$f(x)\xi_{m,n,p}=\frac{(n-m-1)f(x)}{\Gamma(n-m)}\int_{x}^{\beta}\{g(y)-g(x)\}^{p}\left\{ln\left(\frac{\overline{F(x)}}{\overline{F(y)}}\right)\right\}^{n-m-2}\frac{f(y)}{\overline{F(x)}}\,dy$$
$$+\frac{p\,\acute{g}(x)\overline{F(x)}}{\Gamma(n-m)}\int_{x}^{\beta}\{g(y)-g(x)\}^{p-1}\left\{ln\left(\frac{\overline{F(x)}}{\overline{F(y)}}\right)\right\}^{n-m-1}\frac{f(y)}{\overline{F(x)}}\,dy\,.$$

We can obtain (using (16)) that

$$f(x)\xi_{m,n,p}=f(x)\xi_{m+1,n,p}+p\,\acute{g}(x)\overline{F(x)}\xi_{m,n,p-1},$$

and from (14)

$$p\,\acute{g}(x)\xi_{m,n,p-1}=\frac{f(x)}{\overline{F(x)}}\left(\xi_{m,n,p}-\ \xi_{m+1,n,p}\right)=\frac{p\,f(x)}{\overline{F(x)}}\frac{\Gamma(p+n-m-1)}{c^{p}\Gamma(n-m)}=\frac{p\,f(x)}{c\,\overline{F(x)}}\xi_{m,n,p-1},$$

which implies

$$\acute{g}(x) = \frac{f(x)}{c\,\bar{F}(x)}, \tag{18}$$

hence, the Theorem is proved.
When b = 1,which treated by Noor and Athar [19].

CHARACTERIZATIONS

In this subsection, characterizations based on truncated moments of the given function $g(x)$ are presented.

Theorem 3:
Referring to (5) and (7), then

$$E[g(X)|X \le x] = g(x) + \frac{1}{cd}\left\{1 - 2F1\left(1, d; d+1; \frac{[F(x)]^{\frac{1}{d}}}{a}\right\}, \quad \alpha \le x \le \beta, \tag{19}$$

where $c \neq 0, d > 0$ and $2F1(.,.;.;.)$ is a hypergeometric function.

Proof
From (5) and (7), we get

$$\int_{\alpha}^{x} g(u)\, f(u) d\,u = F(x)\left\{g(x) + \frac{1}{cd}\right\} + \frac{a}{bcd}\int_{\alpha}^{x} e^{cg(u)}\, f(u) d\,u. \tag{20}$$

Let $t = [\frac{F(u)}{F(x)}]^{\frac{1}{d}}$, then

$$\int_{\alpha}^{x} e^{cg(u)}\, f(u) d\,u = \frac{-bdF(x)}{a}\int_{0}^{1} t^{d-1}\,[1 - \left(\frac{[F(x)]^{\frac{1}{d}}}{a} t\right)]^{-1}\,\mathrm{dt}$$

$$= \frac{-bdF(x)}{a}\; 2F1(1, d; d+1; \frac{[F(x)]^{\frac{1}{d}}}{a}), \tag{21}$$

where $2F1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_{0}^{1} t^{b-1}(1-t)^{c-b-1}\,(1-z)^{-a} dt$, see

Gradstein and Ryzhyk [10] in Page 995.
Now, from (20) and (21), we get (19).
When $d = 1$, the $F(x) = a + b\, e^{-c\, g(x)}$, which studied by Hamedani et al. [11].
The following Table 1 gives some examples of (7) distributions.

Table 1. *Examples of (7)*

Distribution	$F(x)$	$g(x)$	a	b	c	d
Weibull	$1 - e^{-\theta x^{p}}, \quad x > 0$	x^{p}	1	-1	θ	1
Pareto of the first kind	$1 - \lambda^{p} x^{-p}, x > \lambda$	$ln\ x$	1	$-\lambda^{p}$	p	1
Burr XII	$1 - (1 + \theta\, x^{p})^{-\lambda}, x > 0$	$ln\,(1 + \theta\, x^{p})$	1	-1	λ	1
Rayleigh	$1 - e^{-\theta x^{2}}, \quad x > 0$	x^{2}	1	-1	θ	1
Lomax	$1 - (1 + \theta\, x)^{-\lambda}, x > 0$	$ln\,(1 + \theta x)$	1	-1	λ	1
Inverse Weibull	$e^{-\theta x^{-p}}, \quad x > 0$	x^{-p}	0	1	θ	1
Power function	$\left(\frac{x}{\lambda}\right)^{p}, 0 < x < \lambda$	$ln\left(\frac{x}{\lambda}\right)$	0	1	-p	1

Rectangular	$\left(\frac{x-\beta}{\lambda-\beta}\right), \beta < x < \lambda$	$ln\,(x-\beta)$	0	$\frac{1}{\lambda-\beta}$	-1	1
Cauchy	$\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right), -\infty < x < \infty$	$ln\left(\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)\right)$	$\frac{1}{2}$	$\frac{1}{\pi}$	-1	1
Pareto of the second kind	$1-(1+x)^{-\lambda}, x > 0$	$ln\,(1+x)$	1	-1	λ	1
Exponential	$1-e^{-\lambda x}, x > 0$	x	1	-1	λ	1
Inverse Exponential	$e^{-\frac{\lambda}{x}}, \quad x > 0$	$\frac{1}{x}$	0	1	λ	1
Gumbel	$e^{-e^{-\beta(x-\lambda)}}, -\infty < x < \infty$	$e^{-\beta(x-\lambda)}$	0	1	1	1
Kumaraswamy	$1-(1-x^p)^\lambda, 0 < x < 1$	$ln\,(1-x^p)$	1	-1	$-\lambda$	1
Exponentiated Frechet	$1-(1-e^{-x^{-\alpha}})^\theta, \quad x > 0$	$\ln(1-e^{-x^{-\alpha}})$	1	-1	$-\theta$	1
Exponentiated Gumbel	$\left[1-e^{-e^{-\left(\frac{x-\mu}{\sigma}\right)}}\right]^\alpha, -\infty < x < \infty$	$e^{-\left(\frac{x-\mu}{\sigma}\right)}$	1	-1	1	α
Exponentiated exponential	$(1-e^{-\lambda x})^\theta, \quad x > 0$	$\ln(1+e^{-\lambda x})$	0	1	$-\theta$	1
Dagum	$[1+(\frac{x}{\theta})^{-\beta}]^{-\alpha}, x > 0$	$\log(\frac{x}{\theta})$	1	1	β	$-\alpha$
Log-Logistic	$[1+(\frac{x}{\lambda})^{-p}]^{-1}, x > 0$	$\log(\frac{x}{\lambda})$	1	1	p	-1
Burr X (Exponentiated Rayleigh)	$(1-e^{-\beta x^2})^\alpha, \quad x > 0$	x^2	1	-1	β	α
Exponentiated Weibull	$(1-e^{-\beta x^p})^\alpha, \quad x > 0$	x^p	1	-1	β	α

Theorem 4:

Consider X to be an absolutely continuous random variables with distribution function F(x) and $x \in (\alpha, \beta)$. Referring to (6) and (8), then

$$E[g(X)|X \geq x] = g(x) + \frac{1}{cd}\left\{1 + 2F1(1,d;d+1;\frac{[\overline{F}(x)]^{\frac{1}{d}}}{a}\right\}, \quad \alpha \leq x \leq \beta, \qquad (22)$$

where $c \neq 0, d > 0$ and $2F1(.,.;.;.)$ is a hypergeometric function.

Proof

From (6) and (8), we can get, as before,

$$\int_\alpha^x g(u)\,f(u)du = \overline{F}(x)\left\{g(x) + \frac{1}{cd}\right\} + \frac{a}{bcd}\int_\alpha^x e^{cg(u)}\,f(u)du$$

$$= \overline{F}(x)\left\{g(x) + \frac{1}{cd}\right\} + \frac{b\overline{F}(x)}{a}\; 2F1(1,d;d+1;\; \frac{[\overline{F}(x)]^{\frac{1}{d}}}{a}),$$

then (22) is satisfied.

When $d = 1$, the $\overline{F}(x) = a + b\, e^{-c\,g(x)}$, which studied by Hamedani et al. [11].

The following Table 2 gives examples of (8) distributions.

Table 2. *Examples of (8)*

Distribution	$\overline{F}(x)$	$g(x)$	a	b	c	d
Weibull	$e^{-\theta x^p}, \quad x > 0$	x^p	0	1	θ	1
Pareto of the first kind	$\lambda^p x^{-p}, x > \lambda$	$ln\,\frac{x}{\lambda}$	0	1	p	1
Burr XII	$(1+\theta\,x^p)^{-\lambda}, x > 0$	$ln\;x$	1	θ	$-p$	$-\lambda$
Rayleigh	$e^{-\theta x^2}, \quad x > 0$	x^2	0	1	θ	1
Lomax	$(1+\theta\,x)^{-\lambda}, x > 0$	$\ln x$	1	1	$-\theta$	$-\lambda$

Inverse Weibull	$1-e^{-\theta x^{-p}}, \quad x>0$	x^{-p}	1	-1	θ	1
Power function	$1-\left(\frac{x}{\lambda}\right)^{p}, 0<x<\lambda$	$ln\left(\frac{x}{\lambda}\right)$	1	-1	$-p$	1
Rectangular	$1-\left(\frac{x-\beta}{\lambda-\beta}\right), \beta<x<\lambda$	$ln\,(x-\beta)$	1	1	-1	$\frac{1}{\beta-\lambda}$
Cauchy	$\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right), -\infty<x<\infty$	$ln\left(\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)\right)$	$\frac{1}{2}$	$-\frac{1}{\pi}$	-1	1
Pareto of the second kind	$(1+x)^{-\lambda}, x>0$	$ln\,(1+x)$	0	1	λ	1
Exponential	$e^{-\lambda x}, x>0$	x	0	1	λ	1
Inverse Exponential	$1-e^{-\frac{\lambda}{x}}, \quad x>0$	$\frac{1}{x}$	1	-1	λ	1
Gumbel	$1-e^{-e^{-\beta(x-\lambda)}}, -\infty<x<\infty$	$e^{-\beta(x-\lambda)}$	1	-1	1	1
Kumaraswamy	$(1-x^{p})^{\lambda}, 0<x<1$	$ln(1-x^{p})$	1	-1	$-p$	λ
Exponentiated Frechet	$(1-e^{-x^{-\alpha}})^{\theta}, \quad x>0$	$x^{-\alpha}$	1	-1	1	θ
Exponentiated Gumbel	$1-\left[1-e^{-e^{-\left(\frac{x-\mu}{\sigma}\right)}}\right]^{\alpha}$	$ln\,[1-e^{-\left(\frac{x-\mu}{\sigma}\right)}]$	1	-1	$-\alpha$	1
Exponentiated exponential	$1-(1-e^{-\lambda x})^{\theta}, \quad x>0$	$ln\,[1-e^{-\lambda x}]$	1	-1	$-\theta$	1

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