SOME ACHIEVEMENTS FROM A STATISTICAL QUEUING THEORY RESEARCH

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ABSTRACT

It is intended in this paper, that is the corrected and enlarged version of Ferreira (2016) presented at APLIMAT 2016, to give an overview on some achievements, considered the most important, of a research on queues through a long time period. Infinite servers' queues are mainly considered. A theory of queues overview, single node and in network is presented. And laying on it some important new theoretical results are highlighted. Also practical applications, the most of them uncommon but some very well known, using mainly infinite servers nodes are outlined. In the theoretical results stand out:

- *An algorithm to compute the sojourn time probability distribution function of a customer in a network which nodes are* $M/G/\infty$ *queues, using traffic equations and Laplace transform,*

- *A collection of service times distributions for which the* $M/G/\infty$ *queue busy period length probability distribution has a particularly simple structure, very rare in the busy period lengths domain.*

The practical applications considered in this work, mainly using the $M/G/\infty$ *queue ability for the study of large populations processes, are in:*

- Logistics, Financial problems, Energy problems, Disease problems, Unemployment situations, Compartment models (Biology, Birth-sickness-death Processes, Hierarchical Systems) and Repairs Shop.

Some historical important parameters in queues domain are named and emphasized its role in the applications. Also some problems, still open problems, as the conditions for the existence of product form equilibrium distribution, are touched upon.

In the end it is given a considerable list of references on these subjects.

Keywords: *Queues, networks of queues, infinite servers queues, achievements*

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QUEUES WITH A SINGLE NODE

Be a **Service Centre** at which arrive units, the **customers**, requiring service to other units, the **servers**, with or without difference among customers and servers.

The most challenging situations concerning these systems study, **Statistical Queuing Theory** subject matter, occur when it is assumed that:

- Customers arrivals follow a stochastic process,
- Each server spent time to supply to each customer the required service is a random variable.

Other relevant factors are:

- The number of servers that may be finite or infinite, constant or variable,
- If the number of servers is finite, some customers must wait to be served. The **waiting capacity**, that may be finite or infinite, is the maximum number of customers that may stay in the Service Centre waiting to be served. The **system capacity** is the maximum number of customers, being served or waiting for service, which are allowed to stay in the Service Centre simultaneously. When a customer arrives at a Service Centre with the capacity complete it is considered lost to the system. So the queue systems with finite capacity are systems with **losses**.
- If the number of servers is infinite a customer that arrives finds immediately an available server. So there is no queue in the formal sense of the term. The queue systems with infinite servers are systems with neither waiting nor losses.
- The **queue discipline** is the method as the customers are selected by the servers or vice-versa. Some examples of queue disciplines are:

"First come-first served" (**FCFS**);
"Last in-first out" (**LIFO**);
"First in-first out" (**FIFO**);
"Processor sharing" (**PS**);
"Service in Random Order" (**SIRO**);
"Priority" (**PRI**);
"General Discipline" (**GD**).

The **arrivals process** is usually characterized through the time length between two successive arrivals of customers, at the Service Centre, probability distribution: **the inter-arrivals time**. It may be deterministic or stochastic. There are models where **batch arrivals** are considered: the number of customers, arriving at each instant of the sequence of the arrivals instants, is a random variable R that can assume integer values greater than 1 - see, for instance, Shanbhag (1966). The arrivals process may depend or not on the number of customers present at the Service Centre. Sometimes **refusal** situations are considered: the customer arrives and refuses to enter in the Service Centre because there are too many customers waiting to be served. And also **renounce** situations: the customer is already in the Service Centre and leaves it because it thinks that has waited a too long time.

The **service process** is specified indicating the length of the time that a customer spends being attended by a server probability distribution: **the service time**. Deterministic or stochastic service times are allowable.

A Service Centre which has associated a service process, a waiting capacity and a queue discipline is a **node**. A node with the respective arrival process is a **queue**.

The Kendall notation, see Kendall (1953), for describing queues is **v/w/x/y/z** where

- **v** denotes the arrival process (D, deterministic; M, exponential; E_k, Erlang (k); G, others),
- **w** denotes the service process (D, deterministic; M, exponential; E_k, Erlang (k); G, others),
- **x** denotes the number of servers,
- **y** denotes the system capacity,
- **z** denotes the queue discipline.

If **y** is not mentioned it is supposed to be infinite. If **z** is not mentioned it is supposed to be FCFS.

QUEUES IN NETWORK

A **network of queues** is a collection of nodes, arbitrarily connected by arcs, through which the customers travel instantaneously and

- There is an arrival process associated to each node,
- There is a **commutation process** which commands the various costumers' paths.

The arrivals processes may be composed of **exogenous arrivals**, from the outside of the collection, and of **endogenous arrivals**, from the collection nodes.

A network is **open** if any customer may enter or leave it. A network is **closed** if it has a fixed number of customers that travel from node to node and there are neither arrivals from the outside of the collection nor departures. A network open for some customers and closed for others is said to be **mixed**.

The commutation process rules, for each costumer that abandons a node, which node it can visit then or if it leaves the network. In a network with J nodes, the

matrix $P=\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1J} \\ p_{21} & p_{22} & \cdots & p_{2J} \\ \vdots & \vdots & & \vdots \\ p_{J1} & p_{J2} & \cdots & p_{JJ} \end{bmatrix}$ is the commutation process matrix, being p_{jl} the probability of a customer, after ending its service at node j, go to node l, $j,l=1,2,...,J$. The probability $q_j=1-\sum_{l=1}^{J} p_{jl}$ is the probability that a customer leaves the network from node j, $j=1,2,...,J$.

A network of queues with infinite servers in each node, with Poisson process exogenous arrivals, may be looked like an $M/G/\infty$ queue. The service time is the sojourn time of a customer in the network. Denote S the sojourn time of a costumer in the network and S_j its service time at node j, $j=1,2,...J$. Be $G(t)$ and $G_j(t)$ the S and S_j distribution functions, respectively, $\overline{G}(s)$ and $\overline{G}_j(s)$ the respective Laplace Transforms.

If $\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_J \end{bmatrix}$ is the network exogenous arrival rates vector, where the rate λ_j is the exogenous arrival rate at node *j* and $\sum_{j=1}^{J} \lambda_j = \lambda$, making $\Lambda(s) = \begin{bmatrix} \lambda_1 \overline{G}_1(s) \\ \lambda_2 \overline{G}_2(s) \\ \vdots \\ \lambda_J \overline{G}_J(s) \end{bmatrix}$ and

$P(s) = \begin{bmatrix} p_{11}\overline{G}_1(s) & p_{12}\overline{G}_2(s) & \cdots & p_{1J}\overline{G}_J(s) \\ p_{12}\overline{G}_1(s) & p_{22}\overline{G}_2(s) & \cdots & p_{2J}\overline{G}_J(s) \\ \vdots & \vdots & & \vdots \\ p_{J1}\overline{G}_1(s) & p_{J2}\overline{G}_2(2) & \cdots & p_{JJ}\overline{G}_J(s) \end{bmatrix}$ results $\overline{G}(s) = \lambda^{-1}\Lambda^T(s)(I - P(s))^{-1}(I - P)\mathrm{A}$

where A is a column with J 1`s, for the Laplace Transform service time (Ferreira and Andrade, 2010d).

The networks of queues with infinite servers in each node have interesting applications in

- Logistics

Based on the failures of the transport vehicles, that allows computing important measures of performance. See, for instance, Filipe and Ferreira (2015), Ferreira, Filipe and Coelho (2014), Ferreira (2013), Ferreira and Filipe (2010a, b), Ferreira, Andrade and Filipe (2009) and Ferreira *et al* (2009). Are considered either a single node or networks of M/G/∞ queues where the customers are the devices failures that are supposed to occur according to a Poisson process. The service time is the time elapsed from the instant the failure is detected till the one at which it is completely repaired.

EQUILIBRIUM DISTRIBUTIONS WITH PRODUCT FORM

The first important result in the network of queues area was about **Jackson networks** - an example of open networks - for which efficient **product form equilibrium distribution** exists (Jackson, 1957).

In a product form solution the equilibrium state probabilities are in the form $\pi(\pi_1, x_2, \ldots x_J) = C\pi_1(x_1)\pi_2(x_2)\ldots\pi_J(x_J)$ where C is a normalizing constant chosen to make equilibrium state probabilities sum to 1 and $\pi_i(.)$ represents the equilibrium distribution for queue $i, i = 1,2,\ldots J$.

The **BCMP** (**B**askett, **C**handy, **M**untz and **P**alacios, 1975) **networks** are a generalization of Jackson networks, considering several classes of customers.

For the **Gordon-Newell networks**-that are closed networks - product form equilibrium distribution also exists (Gordon and Newell, 1967).

QUEUING THEORY STOCHASTIC PROCESSES

A **population** is a set of objects that share common characteristics. Often, in practical situations, it is important to study statistically the expansion, or the reduction, of a population in order, eventually, to control it. If *N (t)* is the population size at instant t, the states of a **population process** are the various values that can be assumed by *N (t)* and the probability that *N (t) =n*, n=0, 1, 2,... is denoted $p_n(t)$.

A **birth** is considered to occur when a new member joins the population. There is a **death** when a member leaves the population.

A population process is a **Markov process** if the changing from a state to another, eventually the same, transition probabilities depend only on the initial state and not on the mutations experienced by the process till the arrival at the present state.

The probability distribution that rules the number of births and deaths in a certain time interval, in a Markov process, depends only on the interval length and not on the initial state.

A queue system is a **birth and death process** with a population composed by customers receiving a service or waiting for it. There is a birth when a customer arrives at the Service Centre. There is a death when a customer abandons the Service Centre. The **state of the system** is the customers' number in the Service Centre. The population process is the most important quantity of interest in the study of queues. In particular, it is important the search for its **stationary** distribution. In this situation $p_n(t)$ do not depend on time and is denoted p_n. Usually p_n is obtained computing $\lim_{t\to\infty} p_n(t)$. The $p_n(t)$, depending on time, characterize the queue system **transient behavior**. The probabilities p_n characterize the queue system **stationary state,** also called **equilibrium state.**

Other important quantities, that are the **queuing system performance** measures, are the **waiting time,** also called **queue time**- the time that a customer spends in the system waiting for the service - and the **sojourn time-** the total time that a customer spends in the system: queue time plus service time.

Often it is difficult to obtain treatable formulae for the population process, the waiting time and the sojourn time and even to make an analytic study. So **numerical** and **simulation methods** are intensively used.

Based on the transient and stationary probabilities of infinite servers systems there are interesting applications in:

- Financial problems

The study of the sustainability of a pensions fund. See, Ferreira, Andrade and Filipe (2012) Ferreira and Andrade (2011h) and Figueira and Ferreira (1999).Two infinite servers queues are considered: one with the contributors to the fund, which service time is the time during which they contribute to the fund; the other with the pensioners which service time is the time during which they receive the pension. In both queues there is no distinction between the customer and its server.

- Energy problems

How to deal with motor cars in a situation of scarce energy – for instance in the end of oil reserves. One possibility, see Ferreira, Filipe and Coelho (2008, 2011), is to recycle or dismantle the motor cars. Using a model with $M/G/\infty$

queues, it is possible to show that the rate at which the motor cars are dismantled or recycled, given by the recycling or dismantling time distribution (that here is the service time distribution) hazard rate function must be greater than the rate at which the motor cars dismantling or recycling is required (the arrivals rate). It is also deduced minimum benefit values above which is interesting either dismantling or recycling

TRAFFIC INTENSITY

The **traffic intensity,** ρ , is the most important parameter in queues study. See, for instance, Cox and Smith (1961). It is given by $\rho = \lambda\alpha$ where λ is the customers **arrival rate** and α **the mean service time.**

Little's formula, see also Cox and Smith (1961), is, perhaps, the most popular result in queuing theory. It is a very general formula usable for any queue system that attains the stationary state. It relates the mean number of customers in the system, N, with the mean sojourn time of a customer, W, through the arrival rate, λ

$$N = \lambda W.$$

Pollaczeck-Khinchine formula (Cox and Smith, 1961) is used, for the M/G/1 queue, to evaluate the mean waiting time of a customer in the system: $W_s = \alpha\frac{\rho}{1-\rho}\frac{1+C_s{}^2}{2}$, C_s is the service time coefficient of variation. The mean sojourn time of a customer in the system is then.

BUSY PERIOD

The **busy period** of a queue system begins when a customer arrives there, finding it empty, and ends when a customer leaves the system letting it empty. Along the busy period there is always at least one customer in the system. In any queue system there is a sequence of **idle periods** and busy periods. In systems with Poisson arrivals the idle period length is always exponential. The statistical study of the busy period is always a very difficult task. In general the busy period length is related with the transient behavior (see, for instance, Ferreira and Andrade, 2009a, b). An idle period followed by a busy period is a **busy cycle**.

For a M/G/∞ queue, if the service time distribution function belongs to the collection

$$G(t) = 1 - \frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w \beta(u)du}dw - \left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w \beta(u)du}dw},$$

$$t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1}$$

the busy period length distribution function is

$$B(t) = \left(1-(1-G(0))\left(e^{-\lambda t-\int_0^t \beta(u)du} + \lambda\int_0^t e^{-\lambda w-\int_0^w \beta(u)du} dw\right)\right) *$$

$$* \sum_{n=0}^{\infty} \lambda^n (1-G(0))^n \left(e^{-\lambda t - \int_0^t \beta(u)du} \right)^{*n}, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^\rho - 1}.$$

If $\beta(t) = \beta$ (constant)

$$G(t) = 1 - \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t} - 1\right) + \lambda}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^\rho - 1}$$

and

$$B^\beta(t) = 1 - \frac{\lambda+\beta}{\lambda}\left(1 - e^{-\rho}\right) e^{-e^{-\rho}(\lambda+\beta)t}, t \geq 0,$$

$$-\lambda \leq \beta \leq \frac{\lambda}{e^\rho - 1},$$

a mixture of a degenerate distribution at the origin and an exponential distribution (Ferreira and Andrade, 2009a,b).

The busy period of the $M/G/\infty$ queue may be useful to model socio-economic problems as, for example

- Disease problems

An epidemic situation may be assumed as being a busy period. An idle period is the one at which there is disease absence. If people fall sick according to a Poisson process, an $M/G/\infty$ queue model may be considered in this case. The customers are the sick people and the service time is the time during which they are sick. There is no distinction between the customer and its server. The pandemic situation is analogous in modelling terms in spite of its much greater dimension and geographical spread.

- Unemployment situations

An unemployment period is a busy period, ironically, and an idle period is a full employment period. The customers are the unemployed people and the service time is the time during which they are unemployed. There is also no distinction between the customer and its server.

In these problems modeling it is also necessary to consider the properties of the transient probabilities. See Ferreira and Andrade (2010e).

MORE APPLICATIONS

Statistical Queuing Theory is applied, for example, to intelligent transportation systems, call centres (see Ferreira and Andrade, 2010a), PABXs, telecommunications networks, advanced telecommunications systems and traffic flow. The networks of queues are used to reduce the waiting times in the hospitals. Another example of application of the networks of queues are the

- Compartment models

The applications presented under this subject illustrate the networks of queues great applicability in modeling real systems in several domains. The three models considered now are related to each other and may be placed among the so called “compartment models”. So be a system in which the customers, after entering, move independently by the various **compartments** that compose it. Interpreting it as a network of queues-each compartment one queue-at which each one isolated, behaves as an $M/G/\infty$ queue, see Ferreira, Andrade and Filipe, (2009), its analysis in equilibrium is quite simple and fits very well in a lot of applications.

1. Biology

In biology this model is used to represent the particles movement through the body of an animal. It is supposed that the system has J compartments and that

- The particles arrival process is Poisson at rate τ,

- The particles move, independently, among the various compartments.

It is easy to state, noting that this system is a network of queues with product form equilibrium distribution, that

Theorem

In equilibrium, the number of particles in compartment J is independent of the number of particles in the other compartments and is Poisson distributed with mean $\tau\mu_j^{-1}$ being μ_j^{-1} the mean time that a particle passing through the system spends in compartment *j, j=1,2,...,J*.■

This is a true result whichever is the complexity of the movement of any particle considered individually. For instance: the sojourn time of a particle in a certain compartment may be arbitrarily distributed and depend on its past history, as well as from the next compartment to be visited. The essential condition is that the particles movements are independent.

2. Birth-sickness-death Processes

Also the birth-sickness-death processes have a structure identical to the one of the former model. The key ideas are, now,

-An individual is born and passes through various health states before dying,

-Conditional on their bearing instants, the individuals move independently through the J states of the system,

-The history of an individual, that is: the states that it will cross and its sojourn in each one is chosen in the bearing time having in account an arbitrarily distribution about such life histories. Being n the number of living individuals, call $\tau(n)$ the probability intensity of a bearing. In the former case $\tau(n) = \tau$, for all n.

Theorem

If an equilibrium distribution exists it is given by

$$\pi(n_1, n_2, \ldots, n_J) = B\left(\prod_{l=0}^{n-1}\tau(l)\right)\prod_{j=1}^{J}\frac{{\mu_j}^{-n_j}}{n_j!}$$

where n_j designates the number of individuals in state j and μ_j^{-1} the mean time that an individual spends in state j during its lifetime.■

3. Hierarchical System

This last model may also be useful to represent the movement of an individual in a hierarchical system:

- The various states correspond to positions inside the hierarchical organization,

- The assumption that the individuals move independently of each other avoids that the model consider systems at which the promotions occur to occupy vacancies. An individual will be promoted as soon as it is habilitated.

- The recruiting rate $\tau(n)$ will be, generally, a decreasing function of *n*. In the birth-sickness-death processes it is natural to consider that $\tau(n)$ is increasing with *n*.

Another interesting application:

- Repairs Shop

Be repairs shop such that

- Accepts two types of repairs,

- Has a total of *K* workers,

- K_1 of these workers perform type 1 repairs. K_2 repairs of type 2 and the remaining K_3 both types of repairs.

Suppose that

- The repairs of type 1 and type 2 are requested to the repairs shop according to independent Poisson processes at rates τ_1 and τ_2, respectively,

- When a repair is requested, it is accepted if there is a habilitated worker available to execute it. If not it is lost to the system, -The repairs that are being dealt with in the shop may travel among the various workers if this allows the acceptation of one more repair,

- The time required to make a type *i* repair is arbitrarily distributed with mean μ_i^{-1}, and it is not influenced by the number of workers that execute it, *i* = 1, 2.

The target is to determine which the best triple is (K_1, K_2, K_3). Call $\psi(n_1, n_2)$ a function that assumes the value *1* when $n_1 \leq K_1 + K_3, n_2 \leq K_2 + K_3, n_1 + n_2 \leq K_1 + K_2 + K_3$ and 0 elsewhere, where n_i is the number of type *i* repairs in the shop, *i* = 1, 2. So, evidently according to the shop rules described above, the probability intensity that a type *i* repair that is requested to the shop is accepted when it has already n_j repairs of type *j* accepted is

$$\tau_1 \frac{\psi(n_1 + 1, n_2)}{\psi(n_1, n_2)}, i = 1$$

$$\tau_2 \frac{\psi(n_1, n_2 + 1)}{\psi(n_1, n_2)}, \quad i = 2$$

Then the equilibrium distribution is

$$\pi(n_1, n_2) = B\psi(n_1, n_2) \frac{1}{n_1!} \left(\frac{\tau_1}{\mu_1}\right)^{n_1} \frac{1}{n_2!} \left(\frac{\tau_2}{\mu_2}\right)^{n_2}$$

Observation:

- The whole system behaves as an infinite servers queue with two types of customers. $\psi(n_1, n_2)$, such as was defined, avoids the consideration of n_1 and n_2 repairs values impossible for the repair shop do deal with them,

- It is possible to solve this problem considering an infinite servers system, with constant arrival rate, conditioning to the states allowed by the repair shop rules,

- The determination of *B* may become computationally complicated if the number of allowed states is too big,

- In Kelly (1979) it is shown that the departure flows (including the lost repairs), constituted by type 1 and type 2 repairs are independent Poisson processes,

- This model may be generalized in order to allow I types of repairs and workers that individually are habilitated to execute a subset of these I types of repairs.

Agner Krarup Erlang, a Danish engineer who worked for the Copenhagen Telephone Exchange, published the first paper on queuing theory in 1909 (Erlang, 1909). The famous **Erlang loss formula** (Erlang, 1917) $p_m = \frac{\rho^m}{m!}\left(\sum_{i=0}^{m}\frac{\rho^i}{i!}\right)^{-1}$ is the stationary probability that in the M/M/m/m queue the m servers are occupied (Ferreira and Andrade, 2010a). It is very much used in call-centres management, to evaluate the probability of losing a call, and also in the occupation study of Hospital's facilities, see for instance Ferreira and Filipe (2015).

Leonard Kleinrock, in the early 1960s, performed an important work on queuing theory used in modern packet switching networks (Kleinrock, 1975, 1976).

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