OPTIMIZATION TOOLS IN MANAGEMENT AND FINANCE

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ABSTRACT

Through this paper, the most commonly used in management and finance optimization problems mathematical tools are presented, in a combined coherent way. The respective mathematical fundaments are synthetically outlined and the resolution methods are briefly described, hopping that this text functions as a manual in these matters.

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INTRODUCTION

It is far from any doubt how interesting the optimization problems are for the management activity, particularly in its finance subdomain. Famous examples are the classical problems *of profits maximization*, *costs minimization*, *production maximization and, a little more recently, risks minimization and reliability maximization.* They are a privileged matter in the Operations Research courses and considered very deeply at the Mathematical Analysis courses.

This work goal is to give an organized vision of the various kinds of problems that rise in this field, from the mathematical point of view and, simultaneously, to describe in a summarized way the respective resolution methods.

To deal with this subject in a complete way was not the intention. On the contrary, the target is to make an inventory of the most recurrent problems, in order that this text serves as a manual in this area.

A small bibliography is presented in the end of trying to supply to the reader quick deeper information on this matter.

OPTIMIZATION PROBLEMS

Usually, in a *optimization problem* it is intended to determine the extreme points-maximums and minimums-of a function

$$y = f(x_1, x_{2,}, \dots, x_n)$$

called *objective function*.

The *independent variables* $x_1, x_2, \dots, x_n$ may be connected for one or more *restrictions* with the form

$$g(x_1, x_2, \dots, x_n) = b, \epsilon\mathbb{R}.$$

This form for the restrictions is quite general and allows the consideration of *inequality restrictions.* In fact they can be transformed in equality restrictions adding new variables designated *lack variables.*

In the sequence of this work, some optimization problems will be presented together with the description of the techniques that make their resolution possible, that is: the determination of the points - the values of $x_1, x_2, \dots, x_n$ - that, simultaneously, satisfy the restrictions and make *y* to assume a maximum or minimum value.

FREE OPTIMUM PROBLEMS

Restrictions are not considered now. To solve these problems, there are mainly three kinds of methods that follow in the sequence.

ANALYTICAL METHODS

Their bases are the two following results:

Theorem 1

In order that a differentiable function $f: D \subset \mathbb{R}^n \to \mathbb{R}$ has a local extreme in an interior point $(a_1, a_{2,}, \dots, a_n)$ a necessary condition is that

$$\left(f'_{x_1}\right)_{(a_1,a_2,\dots,a_n)} = 0$$

$$\left(f'_{x_2}\right)_{(a_1,a_2,\dots,a_n)} = 0$$

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$$\left(f'_{x_n}\right)_{(a_1,a_2,\dots,a_n)} = 0. \blacksquare$$

The points that satisfy this condition are called the f *stationary points.* Note that this condition may be written in the form:

$$\overrightarrow{gradf}(a_1, a_{2,}, \dots, a_n) = 0$$

Theorem 2

Being $(a_1, a_{2,}, \dots, a_n)$ a stationary point of $f: D \subset \mathbb{R}^n \to \mathbb{R}$, belonging to the interior of *D,* and k > 1 the order of the first *f* directed derivative identically non-null at $(a_1, a_{2,}, \dots, a_n)$:

- $f(a_1, a_{2,}, \dots, a_n)$ is a minimum (maximum) if *k* is even and $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ is a positive (negative) defined form.
- $f(a_1, a_{2,}, \dots, a_n)$ is neither a maximum nor a minimum if:

a) *k* is odd,

b) *k* is even and $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ is an undefined form,

c) *k* is even and $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ is a semi-defined form, since either there is a singular direction for which the first non-null directed derivative is odd or, being even, has a sign opposite to the one of $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ outside the singular directions.

- Nothing may be concluded, through the directed derivatives, when *k* is even, $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ is semi-defined and in every singular direction, the first non-null directed derivatives are of even order and assume values with the same sign as $f_{\vec{u}}^{(k)}(a_1, a_{2,}, \dots, a_n)$ outside the singular directions.■

Notes:

- Theorem 1 allows identifying the point's candidates to maximum points or minimum points: the stationary points,
- In Theorem 2 inconclusive situations, a function behavior study in the neighborhood of the stationary point allows, often, to conclude something about the nature of the point. That is the so called *local study*.
- To apply Theorem 2 it is necessary to suppose that f has continuous derivatives, in the stationary point neighborhood, till a convenient order.

If $n = 1$ the situation is a little simpler:

- *a* is a stationary point if $f'(a) = 0$.
- Being *f* a function *k* times differentiable in a neighborhood of *a* – with $k \geq 2$- and $f^{(k)}$the first of the non-null derivatives in *a,* if $f^{(k)}$ is continuous in a:

a) *f* has neither maximum nor minimum in *a* if *k* is odd,

b) If k is even $f(a)$ is a local maximum or minimum as $f^{(k)}(a) < 0$ or $f^{(k)}(a) > 0$.■

Theorem 3

In the case of real functions of real variable note that, frequently, the study of the first derivative signal allows conclusions about the stationary point's nature:

- In points at which the function is not differentiable, evidently, it may assume extreme values, concretely:

a) If $f_l^{'}(a) \geq 0$ and $f_r^{'}(a) \leq 0$, $f(a)$ is a maximum,
b) If $f_l^{'}(a) \leq 0$ and $f_r^{'}(a) \geq 0$, $f(a)$ is a minimum.■

Note:

- It is important to refer here the *variational calculus* methods. Evidently they are also analytical methods, but now the target is the optimization of a *functional* and not of a function. For instance consider the problem of minimizing the functional

$$F[y] = \int_a^b f\big(x, y(x), y^{'}(x)\big)dx.$$

That is, the objective is to determine the function *y(x*) that minimizes $F[y]$. The equation

$$f\big(x, y, y^{'}\big) - \frac{d}{dx} f_{y^{'}}^{'}\big(x, y, y^{'}\big) = 0,$$

called *Euler-Lagrange equation* supplies the solution for the problem.■

Quasi-analytical Methods

Only a short reference to the most commonly used:

- Gradient Method

This method is based on that the *f gradient vector* "points" to the direction along which the function grows quicker.

So, begin calculating a value the $f\big(x_1, x_{2,}, \dots, x_n\big)$ gradient in an initial point $\big(a_1, a_{2,}, \dots, a_n\big)$. Then it is executed an *h* amplitude movement in the direction of the gradient and it is computed the value of the objective function, in the obtained point. If this value is greater than the obtained for *f* in the initial point go on moving in the same direction. If not, compute a new gradient vector and follow a new direction.

Go on this process till the whole gradient vector components are null, having so got the maximum.

If the target is to minimize the objective function, follow the gradient opposite direction and proceed exactly in the same way.■

Steepest Ascent Method

Such as in the former method, the objective function gradient is computed in an initial point. But, the direction of the movement is determined by the greatest gradient component. Only the variable corresponding to this component is changed; the whole others remain unchanged. The changing variable is increased or decreased according to the partial derivative signal. After a new point is obtained, the process is repeated till attaining the optimum with the desired approximation.■

Heuristic Methods

These methods are applied when the problem does not lead to an evident mathematical quantification. Here are two examples:

- Bolzano Search Method

It is useful when a function *f(x),* that is not explicitly known, but may be formulated experimentally, is supposed to be convex and admit continuous derivatives in the $[a, b]$ interval.

To apply the method it is necessary to know $f'(x_1) = f'\left(\frac{b-a}{2}\right)$. Then calling x_0 the minimum point wanted:

a) If $f'(x_1) > 0$, search $a \leq x_0 \leq x_1$,
b) If $f'(x_1) < 0$, search $x_1 \leq x_0 \leq b$,
c) If $f'(x_1) = 0$, search $x_1 = x_0$.

In each iteration the search interval amplitude is half reduced. After *N* steps, the search interval amplitude is reduced of a $\left(\frac{1}{2}\right)^{N-1}$ factor.■

- Fibonacci Search Method

It is useful when a function *f(x),* which form is not explicitly known, is supposed to be continuous and concave in the $[a, b]$ interval. And the maximum point $f(x_0)$ is such that $x_0 \in [a, b]$. Begin computing $f(x_1) = f\big(a + F_1(b - a)\big)$ and $f(x_2) = f\big(a + F_2(b - a)\big)$ where $F_1 = .38$ and $F_2 = .62$ are the *Fibonacci numbers.* So,

a) If $f(x_1) < f(x_2)$, $x_0 \in [x_1, b]$,

b) If $f(x_1) > f(x_2)$, $x_0 \in [a, x_2]$,

c) If $f(x_1) = f(x_2)$, either $x_0 \in [a, x_2]$ or $x_0 \in [x_1, b]$.

After *N* iterations the amplitude of the original interval is reduced of a $(.62)^{N-1}$ factor.■

Any of the methods, in its specific field of application, allows obtaining x_0 as closely as desired.

CONSTRAINED OPTIMUM PROBLEMS

The target in this kind of problems is the optimization of a *n* variables function,
$f(x_1, x_{2,}, \ldots, x_n)$, constrained to *m* equality restrictions, *m<n,* $g_i(x_1, x_{2,}, \ldots, x_n) = b_i$, $b_i \in \mathbb{R}, i = 1, 2, \ldots, m$.

To solve this problems is indicated the

Lagrange Method (undetermined multipliers)
Begin to build the *Lagrangean function*

$$\mathcal{L}(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = f(x_1, x_{2,}, \dots, x_n) + \sum_{i=1}^{m} \lambda_i [b_i - g_i(x_1, x_{2,}, \dots, x_n)]$$

where the $\lambda_i's$ are the *Lagrange multipliers*. It is usual to call $x_1, x_{2,}, \dots, x_n$ *decision variables* and $\lambda_1, \lambda_2, \dots, \lambda_m$ *auxiliary variables*. The next step is to determine the $\mathcal{L}$ stationary points, candidates to maximums or minimums that result from the resolution of the system constituted by the equations.

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = 0, j = 1, 2, \dots, n$$

and

$$g_i(x_1, x_{2,}, \dots, x_n) = b_i, i = 1, 2, \dots, m.$$

Finally, for each stationary point it is necessary to decide if it is a maximum or a minimum. For that build the *bordered Hessian*:

$$\overline{H}(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_m) = \begin{vmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \\ \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} & \begin{bmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1^2} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \mathcal{L}}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 \mathcal{L}}{\partial x_n^2} \end{bmatrix} \end{vmatrix}$$

and consider the *n – m principal minors* obtained suppressing the last lines and columns of $\overline{H}$. Order these minors, $H_1, H_2, \dots, H_{n-m}$, being H_1 the one of lesser order and $H_{n-m} = \overline{H}$ the greatest order one. So,

a) The stationary point will be a minimum if

i. m is even and $H_i > 0, i = 1, 2, \dots, n - m$,
ii. m is odd and $H_i < 0, i = 1, 2, \dots, n - m$.

b) The stationary point will be a maximum if

i) *m* is even and $(-1)^i H_i > 0, i = 1, 2, \dots, n - m$,
ii) *m* is odd and $(-1)^i H_i < 0, i = 1, 2, \dots, n - m$.■

Notes:

- Note that there is a guarantee that any obtained solution will be optimal if it can be found an $\mathcal{L}(.)$ optimum.

- If n = m + 1 these problems may be solved transforming them in free optimum problems with *n* = 1, because, after the restrictions, it is possible to write *m* variables as functions of only one that will be the problem independent variable. It is the so called *explicit method*. There is an inconvenient: the Lagrange multipliers are not considered and its interpretation, for management purposes, is important.
- The difference *n* – *m* gives the problem *degrees of freedom* that, as it is logical, is coincident with the number of principal minors to be considered.■

An evident application of this tool is the problem of maximization of a production function constrained by a budget.

MATHEMATICAL PROGRAMING PROBLEMS

For this kind of problems inequality restrictions may be considered and it is imposed that $x_i \geq 0, i = 1, 2, \ldots, n$. Contrarily to what happened in the constrained optimum problems there is no bound for the restrictions number, since they define a set of solutions – also called *opportunities set* – non empty.

A class of convex programming problems, at which it is intended to minimize convex functionals subject to convex inequalities, is outlined now. Begin presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities. Note that it is not necessary to impose any continuity conditions. Only geometric conditions are important.

Theorem 4 (Kuhn-Tucker**)**

Be *f(x)*, $f_i(x), i = 1, \ldots, n$, convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem $\min_{x \in C} f(x)$, $sub.: f_i(x) \leq 0, i = 1, \ldots$. Be x_0 a point where the minimum, supposed finite, is reached. Suppose also that for each vector *u* in E_n, Euclidean space with dimension *n,* non-null and such that $u_k \geq 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*. So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x \in C} \left\{ f(x) + \sum_1^n v_k f_k(x) \right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \ ,$$

ii) For every vector u in E_n with non-negative components, that is: belonging to

the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x) \geq f(x_0) + \sum_1^n v_k f_k(x_0) \geq f(x_0) + \sum_1^n u_k f_k(x_0) \ . ■$$

Corollary 4 (Lagrange duality**)**

In the conditions of Theorem 4$f(x_0) = \sup_{u \geq 0} \inf_{x \in C} f(x) + \sum_1^n u_k f_k(x)$. ■

Notes:

- -This corollary is useful supplying a process to determine the problem optimal solution,
- -If the whole v_k in expression at *i)* are positive, x_0 is a point that belongs to the border of the convex set defined by the inequalities,
- -If the whole $v_k$are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the restrictions free problem.■

Considering non-finite inequalities:

Theorem 5 (Kuhn-Tucker in infinite dimension**)**

Be *C* a convex subset of a Hilbert space *H* and *f* (*x*) a real convex functional defined in *C*. Be *I* a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and *F*(*x*) a convex transformation from *H* to *I* (convex in relation to the order introduced by cone $\wp$: if $x, y \in \wp, x \geq y \; if \; x - y \in \wp$). Be x_0 a *f* (*x*) minimizing in *C* subjected to the inequality $F(x) \leq 0$.Consider $\wp^* = \left\{x: [x, p] \geq 0, \begin{matrix} \forall \\ x \in \wp \end{matrix}\right\}$ (dual cone). Admit that given any $u \in \wp^*$ it is possible to determine *x* in *C* such that $[u, F(x)] < 0$. So, there is an element *v* in the dual cone $\wp^*$, such that for *x* in *C* $f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)]$, being *u* any element of $\wp^*$. ■

Corollary 5 (Lagrange duality in infinite dimension**)**

$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)])$ in the conditions of Theorem 5.■

Theorem 4 and Theorem 5 describe the Kuhn-Tucker Method that can be operationalized as follows:

Kuhn-Tucker Method

It is an extension of Lagrange method seen above. Be an objective function $f(x_1, x_{2,}, \ldots, x_n)$ constrained by *m* restrictions: *k* of the form $g_i(x_1, x_{2,}, \ldots, x_n) \leq b_i, i = 1, 2, \ldots, k$, and *m* – *k* of the form $g_i(x_1, x_{2,}, \ldots, x_n) \geq b_i, i = k + 1, \ldots, m$. Now the Lagrangean has the form

$$\mathcal{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_m) =$$

$$= f(x_1, \ldots, x_n) - \sum_{i=1}^{k} \lambda_i [b_i - g_i(x_1, \ldots, x_n)] - \sum_{i=k+1}^{m} \mu_i [b_i - g_i(x_1, \ldots, x_n)].$$

The conditions to be satisfied for a candidate to an optimum are

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{k} \lambda_i \frac{\partial g_i}{\partial x_j} + \sum_{i=k+1}^{m} \mu_i \frac{\partial g_i}{\partial x_j} = 0\,, j = 1, 2, \dots, n$$

$$\lambda_i [b_i - g_i(x_1, \dots, x_n)] = 0, i = 1, 2, \dots, k$$

$$\mu_i [b_i - g_i(x_1, \dots, x_n)] = 0, i = k + 1, \dots, m$$

$$g_i(x_1, \dots, x_n) \le b_i, i = 1, 2, \dots, k$$
and ■
$$g_i(x_1, \dots, x_n) \ge b_i, i = k + 1, \dots, m$$

Generally, the *Kuhn-Tucker conditions* supply a set of necessary conditions for a point to be an optimum point.

A point that satisfies the *Kuhn-Tucker conditions* may be a global optimum, a local optimum or neither one nor other, depending on the objective function and restrictions convexity or concavity. But, if $f(x_1, x_{2,}, \dots, x_n)$ is concave (convex) and the restrictions define a convex (concave) set, the Kuhn-Tucker theory establishes that any point satisfying the necessary conditions seen before is a local optimum.

Notes:

- A mathematical programming problem is *linear* if $f(x_1, x_{2,}, \dots, x_n)$ and $g_i(x_1, x_{2,}, \dots, x_n), i = 1, 2, \dots, m$ are linear functions of the respective arguments. A method to solve this kind of problems is the *Simplex Method*, at which it is used a methodology identical to the one of the Steepest Ascent Method.
- An *integer programming* problem is a *linear programming* problem at which the values of the variables are integer numbers. Among the useful algorithms to solve this kind of problem are *branch and bound* and *cut* algorithms. From these last ones the most used is *Gomory* algorithm.
- As integer programming problems particular cases there are *transportation* problems and *allocation* problems. In the resolution of the first ones it is used the *transportation algorithm*. The allocation problems may also be formalized as transportation problems. But it is not usual to make such a procedure and instead it is used in its resolution the method known as the *Hungarian Method*.
- A *quadratic programming* problem is a mathematical programming problem at which the whole restrictions are linear and the objective function has the form

$$f(x_1, x_{2,}, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i$$

being c_{ij} and d_i known constants. For its resolution may be applied the Kuhn-Tucker method, as in the whole problems that are being inventoried, but the most used is the *Frank and Wolfe Method*, typical of these problems. ■

Quadratic programming is applied in portfolios management.

When presenting the Bolzano and Fibonacci search methods, solutions for the optimization problem of $f(x)$ being $a \leq x \leq b$ were looked for. It may be said that a programming problem, nonlinear, mono-variable was being solved.

Zener-Duffin Method

Usually it is briefly described as a *geometric programming* method. It is applied in the optimization of functions with the form

$$f(x_1, x_{2,}, \dots, x_n) = \sum_{i=1}^{m} T_i$$

where

$$T_i = a_i \prod_{j=1}^{n} x_j^{b_{ij}}$$

being a_i and b_{ij} real numbers.

Zener established that if $m = n + 1$, to optimize *f* is necessary to find adequate constants, c_j, such that

$$\prod_{j=1}^{n} T_i^{c_j} = K, K \text{ constant}$$

and

$$\sum_{j=1}^{n} c_j = 1.$$

So, the optimal value is given by

$$f_{opt} = \frac{K}{\prod_{j=1}^{n} {c_j}^{c_j}}. \blacksquare$$

Duffin extended the work of Zener in the way not to be necessarily *m* = *n* +1. In that case it is obtained a lower bound and an upper bound for f_{opt}.

Dynamic Programming

Its objective is the optimization of functions in the form

$$f(x_1, x_{2,}, \dots, x_n) = \sum_{i=1}^{n} f_i(x_i) \text{ with } \sum_{i=1}^{n} x_i = K, K \in \mathbb{R}$$

The dynamic programming is based on *Bellman principle:*

- So that a politic is optimal it is necessary that anyone the initial state is and anyone the initial decision taken is, each following decision must be optimal when related to the resulting state of the decision taken immediately before.

Using the dynamic programming it is possible to transform the initial problem in a sequence of *n* problems, in the form of a recursive relation:

$$m_n(K) = \underset{0 \leq x \leq K}{\text{opt}} \{f_n(x)\}$$

$$m_j(K) = \underset{0 \leq x \leq K}{\text{opt}} \left\{f_j(x) + m_{j+1}(K + x)\right\}, j = n - 1, n - 2, \dots, 1$$

The optimum so obtained may be maximum or minimum according to the nature of the functions f_i.■

MINIMAX THEOREM

Consider then the zero-sum two players games formulation:

- Be $\phi(x, y)$ a real function of two real variables $x, y \in H$ (real Hilbert space),
- Be *A* and *B* two convex sets in *H*,
- One of the players chooses strategies (points) in *A*, in order to maximize $\phi(x, y)$ (or to minimize $(-1)\phi(x, y)$): it is the *maximizing* player,
- The other player chooses strategies (points) in *B*, in order to minimize $\phi(x, y)$ (or to maximize $(-1)\phi(x, y)$): it is the *minimizing* player.
- $\phi(x, y)$ is the payoff function. The value $\phi(x_0, y_0)$ represents, simultaneously, the maximizing player earning and the minimizing player loss in a move at which they chose, respectively, the strategies x_0 and y_0.

This game has a value *G* if

$$\sup_{x \in A} \inf_{y \in B} \phi(x, y) = G = \inf_{y \in B} \sup_{x \in A} \phi(x, y) .$$

If for some (x_0, y_0), $\phi(x_0, y_0) = G$, (x_0, y_0) is a pair of optimal strategies. It is also a saddle point if

$$\phi(x, y_0) \leq \phi(x_0, y_0) \leq \phi(x_0, y), x \in A, y \in B.$$

Theorem 6

Consider *A* and *B* closed convex sets in *H*, with *A* bounded. Be $\phi(x, y)$ a real functional defined for *x* in *A* and *y* in B such that:

- $\phi(x, (1 - \theta)y_1 + \theta y_2) \leq (1 - \theta)\, \phi(x, y_1) + \theta\phi(x, y_2)$ for *x* in *A* and y_1, y_2 in *B*, $0 \leq \theta \leq 1$,

- $\phi\big((1-\theta)x_1 + \theta x_2, y\big) \geq (1-\theta)\,\phi(x_1, y) + \theta\phi(x_2, y)$ for *y* in *B* and x_1, x in *A*, $0 \leq \theta \leq 1$,
- $\phi(x, y)$ is continuous in *x* for each *y*, then the game has a value.■

The *Minimax theorem*, from von Neumann, is obtained as a corollary of Theorem 6, strengthening its hypothesis:

Theorem 7 (Minimax**)**

Suppose that the Theorem 6 functional $\phi(x, y)$ is continuous in both variables, separately, and is also bounded. Then there is an optimal pair of strategies fulfilling the property of being a saddle point. ■

Consider a zero-sum two players game. Calling *A* the *maximizing* player and *B* the *minimizing* player, the payoff table when *A* chooses the strategy *i*, *i* = 1, 2, ..., *m* and *B* the strategy *j*, *j* = 1, 2, ..., *n* is

$$\begin{array}{cc} & Player\ B \\ Player\ A & \left[g_{ij}\right] \end{array} \qquad i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

reading the player *A* the values as gains and the player *B* as loses. Of course a negative gain is a loss and *vice-versa*. In the terms of von Neumann Minimax Theorem the problem may be solved as a linear programming problem:

For player *A*

The target is to maximize G (value of the game) subject to the constraints

$$g_{11}x_1 + g_{21}x_2 + \ldots + g_{m1}x_m \geq G$$
$$\vdots$$
$$g_{1n}x_1 + g_{2n}x_2 + \ldots + g_{mn}x_m \geq G$$

$$x_1 + x_2 + \ldots + x_m = 1$$
$$x_1, x_2, \ldots, x_m \geq 0$$

being x_i the frequency at which the player chooses its *i* strategy, *i* = 1, 2, ..., *m*.

For player *B*

The target is to minimize H (value of the game) subject to the constraints

$$g_{11}y_1 + g_{12}y_2 + \ldots + g_{1n}y_n \leq H$$
$$\vdots$$
$$g_{m1}y_1 + g_{m2}y_2 + \ldots + g_{mn}y_n \leq H$$

$$y_1 + y_2 + \ldots + y_n = 1$$
$$y_1, y_2, \ldots, y_n \geq 0$$

being y_j the frequency at which the player chooses its *j* strategy, *j* = 1, 2, ..., *n*.

When there is a solution $G = H$.

If

$$\max_i \min_j [g_{ij}] = \min_j \max_i [g_{ij}], i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

the common value is the value of the game and the mathematical programs presented above are avoidable.■

Note:

- In this kind of problems there is simultaneously a maximization of minimums and a minimization of maximums.

NASH THEOREM

For the case of non-zero-sum games involving two or more players in direct competition – non-cooperative games – it is not possible to use the Minimax theorem as it was shown above. Instead it is useful a Minimax theorem generalization from John Nash:

Theorem 8 (Nash)

Any non-cooperative game of *n* players, in which each player has a finite number of strategies, has at least one set of equilibrium strategies.■

Note:

- -This theorem shows that there can be multiple equilibrium strategies
- -Despite being non-cooperative games, the theorem shows that players earn more if they agree to cooperate. ■

Nash Equilibrium

It is a game theory solution concept of non-cooperative games with two or more players, in which each player is assumed to know the other players equilibrium strategies, and no player has anything to gain by changing only their own strategy.■

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