A TOPOLOGICAL APPROACH TO CONSUMER THEORY

Manuel Alberto M. Ferreira

*Instituto Universitário de Lisboa (ISCTE-IUL),
BRU-IUL, ISTAR-IUL, Lisboa, **PORTUGAL***

E-mail: manuel.ferreira@iscte.pt

ABSTRACT

The presentation of an interesting application of mathematics in economics, more concretely of topological notions in economics: the formulation of the theory of consumer basic problem built on topology, is the main target of this work. This is made grounded on the concept of preferences relation and operationalized with optimization tools.

Keywords: *Preferences relation, optimization, consumer theory, demand*

INTRODUCTION

This work target is to present a mathematical formulation of the consumer theory, as rigorous as possible, using mostly topological notions. This is done using mainly the concept of preferences relation and then optimization mathematical tools.

So, in the next section, the preferences relation concept and the respective axioms are presented and the model to be used is outlined.

Then it is established the consumer theory and some important consequent results are shown.

Finally, the paper ends with a section of conclusions and bibliography on these subjects.

INTRODUCING PREFERENCES RELATIONS

To use mathematical tools to deal with the concept of consumer basket, consider a non- empty convex closed subset X in $\mathbb{R}_+{}^n$ where is defined a preferences relation, symbolized $\succcurlyeq$, as follows:

Definition 2.1
For any x, y and z belonging to X,
i) $x \succcurlyeq y \ \vee \ y \succcurlyeq x$ (Completeness),
ii) $x \succcurlyeq y \ \wedge \ y \succcurlyeq z \implies x \succcurlyeq z$ (Transitivity)
iii) For any $y \in X$, $\{x \in X : x \succcurlyeq y\}$ and $\{x \in X : y \succcurlyeq x\}$ are closed subsets (Continuity),

iv) $x \geq y$ (that   is:$x_i \geq y_i, i = 1,2,..n$)   and $x \neq y \Rightarrow x > y$ (Strong Monotonicity).■

Note:

- $x \succ y$ if $x \succcurlyeq y$ and $y \succcurlyeq x$ is false,

- In economics language X is the set of a consumer possible baskets and $\succcurlyeq$ represents that consumer preferences lengthwise this set of baskets.

Definition 2.2

$x, y \in X$ are indifferent, and it is denoted $x \sim y$, when $x \succcurlyeq y$ and $y \succcurlyeq x$.■

Definition 2.3[1]

A basket $x \in X$ is said to be redundant when there is $y \in X$ such that $x \geq y, y \neq x$, and $x \sim y$. ■

To present some results, to the Definition 1.1 must be added the following two axioms:

v) $x, y \in X, x \neq y$ and x, y not redundant such that $x \sim y \Rightarrow (1 - t)x + ty > x, \forall t \in [0,1]$.■

Note:

-That is: the convex linear combination of two different, not redundant and indifferent baskets is preferred to one of them isolated.

vi) $\forall x \in X, \forall \varepsilon > 0, \exists y \in X$ such that $\|x - y\| < \varepsilon$ and $y > x$.■

Note:

-That is: in any neighborhood of a basket there is at least one other basket preferable to it.

Theorem 2.1

Supposing that $\succcurlyeq$ fulfills the axioms *i), ii), iii)* and *iv)* in Definition 2.1, if $X = \mathbb{R}_+^n$ there is a continuous function $u: X \to \mathbb{R}$ such that $u(x) \geq u(y)$ if and only if $x \succcurlyeq y, \forall x, y \in X$.

Dem:

Call **1** the vector with the whole n coordinates equal to 1. For $x \in X$ define $A = \{t \in \mathbb{R}_+ : t\mathbf{1} \succcurlyeq x\}$ and $B = \{t \in \mathbb{R}_+ : x \succcurlyeq t\mathbf{1}\}$. Per *iv)* A and B are nonempty and by *iii)* are both closed since the function $\varphi: \begin{array}{c} \mathbb{R}_+ \to \mathbb{R}_+^n \\ t \to t\mathbf{1} \end{array}$ is continuous and, here, $A = \varphi^{-1}(\{y \in X : y \succcurlyeq x\})$ and $B = \varphi^{-1}(\{y \in X : x \succcurlyeq y\})$. After *i)* it results $A \cup B = \mathbb{R}_+$ and by the $\mathbb{R}_+$ connexity [2] there is $t \in \mathbb{R}$ fulfilling $t\mathbf{1} \sim$ x**.** Again, by *iv)* this t is unique. So define $u: X \to \mathbb{R}$ as $u(x) = t$. Besides $u^{-1}([t_0, \infty]) = \{x \in X : u(x) \geq$

[1]See Simonsen (1989).

[2] A set is connex if it cannot be represented as the union of two separable sets both non-empty.

$t_0\} = \{x \in X: x \succcurlyeq u^{-1}(t_0)\}$ and $u^{-1}([0, t_0]) = \{x \in X: u^{-1}(t_0) \succcurlyeq x\}$ are closed and then it results that u is continuous[3].∎

Note:

-More generally it is also true that "If $X \subset \mathbb{R}^n$ is a connex set and $\succcurlyeq$ is a preferences relation in X fulfilling *i), ii)* and *iii)*, there is a continuous function $u: X \to \mathbb{R}$ such that $u(x) \geq u(y) \Leftrightarrow x \succcurlyeq y, \forall x, y \in X$", see Debreu (1959),

-This theorem shows that under axioms *i), ii), iii)* and *iv)* it is possible to determine a numerical scale for the consumer preferences what is extremely important in Decision Theory.

ESTABLISHING CONSUMER THEORY

Begin by a formulation, among many equivalent, of the theory of consumer basic problem:

-Call r the consumer income and $p = (p_1, p_2, \ldots, p_n)$ the n goods vector prices. If $u: X \to \mathbb{R}$ represents the consumer preferences, the preferences maximization problem can be mathematically formulated as

$$\begin{aligned} \max \ & u(x) \\ s.to \ & \langle p, x \rangle \leq r \end{aligned} \quad (3.1).$$

It is supposed that

$$A = \{x \in X: \langle p, x \rangle \leq r\},$$

the opportunities set, is non-empty.∎

Note:

-Being u continuous and $p \in \mathbb{R}_{++}^n$, so being A compact, the Weierstrass theorem guaranties that (3.1) is possible.

Definition 3.1

The indirect utility function $v(p, r)$ is defined through the problem:

$$\begin{aligned} v(p, r) = \max \ & u(x) \\ s.to \ & \langle p, x \rangle \leq r, x \in X \end{aligned} \quad (3.2). \ ∎$$

Note:

-It is easy to check that if x is a solution for (3.2) $\langle p, x \rangle = r$ since either *vi)* or *iv)* are satisfied for $X = \mathbb{R}_+^n$,

- x is not redundant,

- If, in addition, *v)* is satisfied x is unique.

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[3]Because in this case the whole closed sets in $[0, \infty)$ will have inverse image closed. For details see (Cysne and Moreira, 1997), page 95.

Definition 3.2

The vector only solution x for the problem (3.2), given p and r: $x(p, r)$ is the marshallian demand.∎

Note:

-$x_i(p, r)$ is the marshallian demand for the good i, $i = 1, 2, \ldots, n$.

Theorem 3.1

If a consumer preferences satisfy *i)*, *ii)*, *iii)* and *vi)*, the marshallian demand $x_i: \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous $\forall i = 1, 2, \ldots, n$.

Dem (See Simonsen (1989)):

Be $(p_n, r_n) \in \mathbb{R}^{n+1}_{++}$ and $(p_n, r_n) \longrightarrow (p, r) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Being easy to check that the sequence $\left(x(p_n, r_n) \right)_{n \in \mathbb{N}}$ is bounded, to evidence that the marshallian demand is continuous, it is enough to evidence that any $\left(x(p_n, r_n) \right)_{n \in \mathbb{N}}$ convergent subsequence converges to $x(p, r)$. So consider the convergent subsequence $\left(x(p_{n_i}, r_{n_i}) \right)_{i \in \mathbb{N}}$ converging to y. As $\langle p_{n_i}, x(p_{n_i}, r_{n_i}) \rangle = r_{n_i}$, in the limit $\langle p, y \rangle = r$, that is: y is feasible in what respects (p, r). To evidence that $y = x(p, r)$ it is enough to evidence that if y' is feasible in what respects (p, r), $u(y) \geq u(y')$. Then define $\mu_n = \frac{\langle p, y' \rangle}{\langle p_n, y' \rangle} \frac{r_n}{r}$. It is immediate to check that $\langle p_n, \mu_n y' \rangle = \frac{\langle p, y' \rangle}{r} r_n \leq r_n$. This means that $\mu_n y'$ is feasible in what respects (p_n, r_n). So, as $x(p_n, r_n)$ is the consumer equilibrium point in what respects the pair (p_n, r_n), $u(x(p_n, r_n)) \geq u(\mu_n y')$ and so going to the limit when $i \to \infty$, remarking that $\mu_n \to 1$ when $n \to \infty$, it is obtained $u(y) \geq u(y')$ so completing the proof.∎

CONCLUDING REMARKS

In this paper, economic concepts were presented through a rigorous formulation, allowing that some economic results were demonstrated in the mathematical sense. Plays here a fundamental role the concept of preferences relation that is axiomatically formulated.

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